Calhoun
Institutional Archive of the Naval Postgraduate School

Calhoun: The NPS Institutional Archive
DSpace Repository

Theses and Dissertations 1. Thesis and Dissertation Collection, all items

1978-09

Analytical hazard representations for use in reliability, mortality and simulation studies

Acar, Mustafa

http://hdl.handle.net/10945/18350

Downloaded from NPS Archive: Calhoun

DUDLEY
KNOX
LIBRARY

Calhoun is the Naval Postgraduate School's public access digital repository for research materials and institutional publications created by the NPS community. Calhoun is named for Professor of Mathematics Guy K. Calhoun, NPS's first appointed -- and published -- scholarly author.

Dudley Knox Library / Naval Postgraduate School
411 Dyer Road / 1 University Circle
Monterey, California USA 93943

http://www.nps.edu/library

ANALYTICAL HAZARD REPRESENTATIONS FOR USE IN RELIABILITY, MORTALITY AND SIMULATION STUDIES

Mustafa Acar

NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

ANALYTICAL HAZARD REPRESENTATIONS FOR USE IN RELIABILITY, MORTALITY AND SIMULATION STUDIES

by

Mustafa Acar

September 1978

Thesis Advisor: D.P. Gaver

Approved for public release; distribution unlimited.

T185337

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS
BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE *(and Subtitle)* Analytical Hazard Representations for Use in Reliability, Mortality and Simulation Studies		5. TYPE OF REPORT & PERIOD COVERED Master's Thesis September 1978
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR*(s)* Mustafa Acar		8. CONTRACT OR GRANT NUMBER*(s)*
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		12. REPORT DATE September 1978
		13. NUMBER OF PAGES 85
14. MONITORING AGENCY NAME & ADDRESS*(if different from Controlling Office)* Naval Postgraduate School Monterey, California 93940		15. SECURITY CLASS. *(of this report)* Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Reliability, Failure rate, Bath tub hazard, Hazard function, Mortality, Nonlinear least-squares, Simulation.

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

A variety of simple analytical models for increasing, decreasing and "bath tub"-type failure rates are discussed. The purpose of this thesis is to develop analytical hazard representations for use in reliability and maintainability studies, and to evaluate them in use for data analysis. Verification of the model was accomplished by computer simulation. They were applied to human mortality and other failure time data.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601 |

Approved for public release; distribution unlimited.

ANALYTICAL HAZARD REPRESENTATIONS FOR USE IN RELIABILITY, MORTALITY AND SIMULATION STUDIES

by

Mustafa Acar
Lieutenant, Junior Grade, Turkish Navy

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the
NAVAL POSTGRADUATE SCHOOL
September 1978

ABSTRACT

A variety of simple analytical models for increasing, decreasing and "bath tub"-type failure rates are discussed. The purpose of this thesis is to develop analytical hazard representations for use in reliability and maintainability studies, and to evaluate them in use for data analysis. Verification of the model was accomplished by computer simulation. They were applied to human mortality and other failure time data.

TABLE OF CONTENTS

I. INTRODUCTION AND SUMMARY 7
  A. INTRODUCTION 7
  B. SYSTEM FAILURE PATTERNS 8

II. ANALYTICAL HAZARD REPRESENTATIONS 12
  A. MODELS FOR THE HAZARD FUNCTION 12
    1. A Bath Tub Model 13
    2. A Decreasing Failure Rate Model 15
  B. MATHEMATICAL PROPERTIES OF THE "BATH TUB" HAZARD MODEL 16
    1. Monotonicity; Quantiles 16
    2. Hazard and Density Function Relationships 18
    3. An Explicit Formula for a Hazard 21
    4. An Explicit Formula for the Failure Time Distribution 22
  C. MATHEMATICAL PROPERTIES OF THE DECREASING FAILURE RATE MODEL 23
    1. The Hazard Behavior 23
    2. Explicit Formulas for the Hazard and the Distribution Function 24
  D. AN ALTERNATIVE "BATHTUB" HAZARD REPRESENTATION 25

III. OBTAINING SPECIFIED HAZARD BEHAVIOR BY SIMPLE SAMPLING 28

IV. COMPUTER SIMULATION AND ESTIMATION PROCEDURE.. 31
  A. SIMULATION AND NUMERICAL RESULTS 31
    1. Algorithm 31
    2. Some Comments 31
  B. EMPIRICAL DENSITY FUNCTION AND ESTIMATION PROCEDURE 35
    1. Empirical Density Function 35
    2. Estimation Procedure for Model Procedures 37
      a. 3-Percentile Approach 37
      b. Nonlinear Least Squares Approach.. 41
  C. TEST PROCEDURE FOR PARAMETERS 61

V. NUMERICAL APPLICATIONS TO TWO SETS OF REAL DATA 64

A. ORAL IRRIGATOR 64

B. HUMAN LIFE (MORTALITY) DATA 66

1. Estimation By Using 3-Percentile Approach 66

2. Estimation By Using Nonlinear Least Squares, and Application of a Nonlinear Programming Algorithm 70

3. Model Modifications 75

C. CONCLUSIONS 83

LIST OF REFERENCES 84

INITIAL DISTRIBUTION LIST 85

ACKNOWLEDGEMENT

The author would like to acknowledge the great help, guidance and advice of Professor D. P. Gaver of the Naval Postgraduate School. The author also wishes to thank Rosemarie Stampfel, typist of the Operations Research curriculum, who typed this thesis.

I. INTRODUCTION AND SUMMARY

A. INTRODUCTION

The failure rate function, or hazard function (hazard for short) may be described as the conditional probability of an equipment's failing at operating age t, having survived to that age. The reliabilities of a variety of electronic and mechanical items are conveniently and naturally described in terms of the appropriate hazard function, and so is the longevity of human beings. The term force of mortality replaces hazard in the latter context.

This paper is devoted to a study of several simple analytical representations for hazard functions. These representations are in turn based upon representations of random variables having certain required properties, in terms of others having familiar distributions--in particular the exponential. Similar ideas are due to Tukey [1], and recently have been examined by Parzen [2]. The hazard representations proposed are quite expeditiously used in simulation studies, e.g. of system reliability or availability in terms of component lifetimes. They may also be used in data analysis studies, in order to parsimoniously describe data sets in terms of perturbations of convenient and familiar standard distributions. Their use in data analysis and simulation is also described in Gaver, Lavenberg, and Price [3], and in Gaver and Chu [4].

B. SYSTEM FAILURE PATTERNS

It is plausible to think that the time series of failures in a system may involve these stages.

Early failures. There may be a relatively large number of failures soon after a system is introduced because of design defects, production errors, or errors stemming from maintenance personnel inexperience. This situation is characterized by a hazard function that is initially large, but that decreases with time. "Infant mortality" is in evidence.

Random Failures. Following the early failure period there may be a period during which failures occur at an essentially constant rate for a rather prolonged time. During this period the hazard function is nearly constant, so the times between failures are close to being exponentially distributed. The effect of age or wearout is not yet apparent.

Wearout Failures. Eventually following the period during which a constant hazard is evident there is likely to be a period of ever-increasing failure rate caused by wearout of system components.

A graphical representation of a hazard function that exhibits the behavior described is given below. Note that it has the legendary "bath tub" shape.

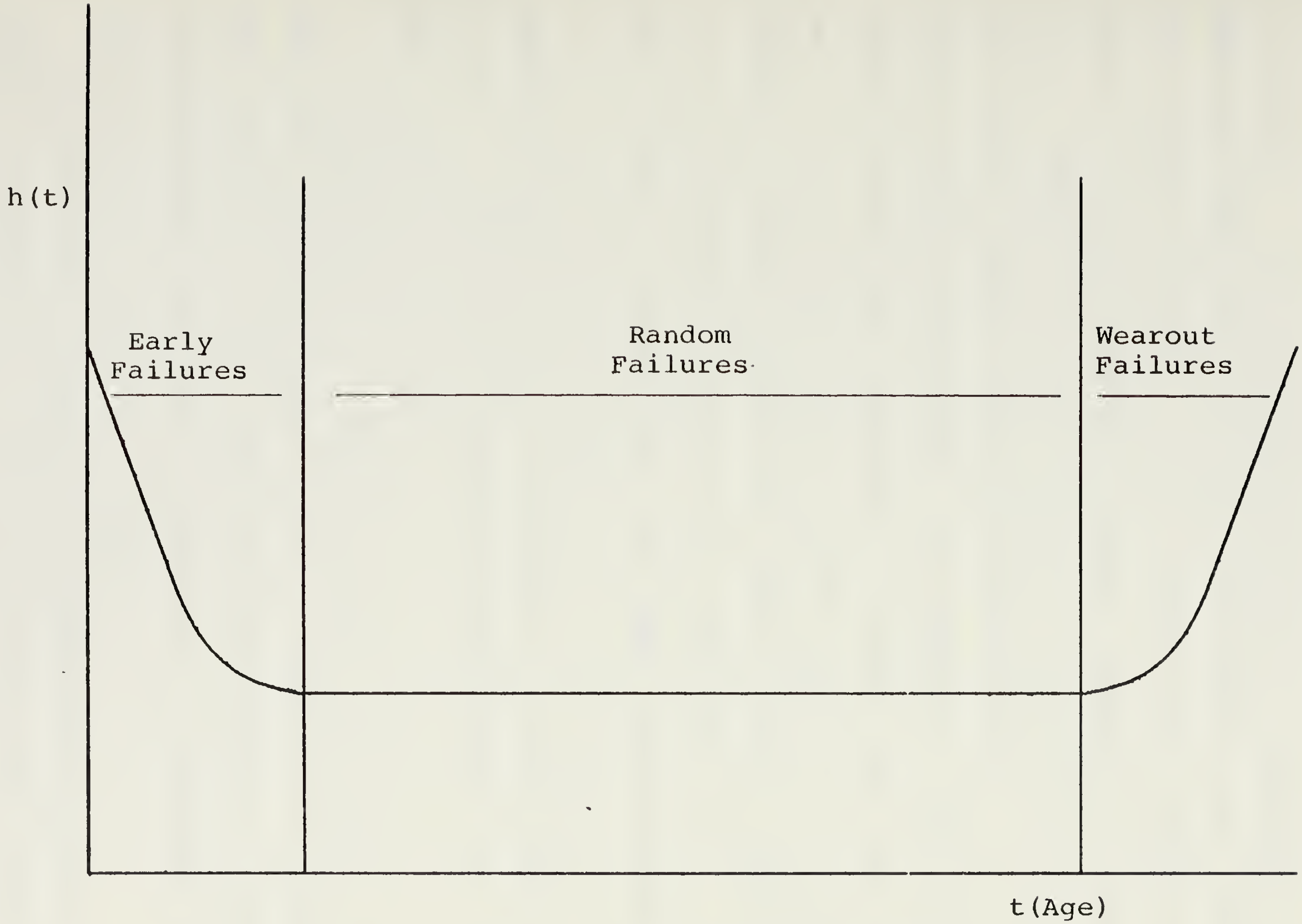

Fig. 1. Bath Tub Shape Curve

Some comments on the above follow:

The term "failure" may refer to an event that is analogous to human death, after which the entire system is replaced. On the other hand repair or component replacement may occur after failure: the system is only repaired, not entirely replaced. In the former case, a hazard function of the kind depicted in Figure 1 applies to each system event ("death"); when the system is installed (or is born), that hazard operates starting from scratch at $t = 0$ until system failure (human death, for instance), after which a similar hazard goes into effect, starting once again from zero. In the latter case, in which repair of a component occurs, a hazard function like that of Figure 1 applies at $t = 0$, but after the first event ("failure") at t_1 a repair action is accomplished. The same hazard operates for $t \geq t_1$ until the next event at $t_2 > t_1$, and so on. Intermediate situations may be envisioned, in which after event n at t_n the hazard governing system failure $n+1$ starts at $t_n - \tau_n$, $0 < \tau_n < t_n$.

Although there is reason to assume that hazards somewhat like that of Figure 1 occur in general for systems, the possibility exists that the system hazard is "bumpy" because wearout failures of components or subsystems may well occur at intermediate times.

If the theory is applied to systems with little or no wearout propensity, as should be the case when dealing with computer software modules, then the hazard function may

well exhibit the initial falloff of Figure 1 but not the rise at later times. In fact, a constant decline as bugs are found and removed could be (optimistically) anticipated for software. The right-hand side of the bath tub vanishes, and the picture is that of a ski slope.

II. ANALYTICAL HAZARD REPRESENTATIONS

A. MODELS FOR THE HAZARD FUNCTION

In this section mathematical models are presented for the failure rate or hazard function. Recall that the hazard may be defined as follows.

Definition. Suppose that the time to failure, X, is a random variable with distribution function $F(x)$, where $F(0) = 0$; the latter possesses the density function $f(x)$, $f(x) = dF/dx$, such that for any positive x,

$$F(x) = \int_0^x f(y)\, dy \,. \tag{2.1}$$

Then the hazard function, or failure rate at age x, is given by

$$h(x) = \frac{f(x)}{1 - F(x)} \,. \tag{2.2}$$

The interpretation of $h(x)dx$ is that it is the conditional probability of failure in the interval $(x, x+dx)$, given that there has been no failure up to age x.

Express the hazard as

$$h(x) = \frac{dF/dx}{1 - F(x)}$$

$$h(x)\,dx = \frac{dF}{1 - F(x)} = -\, d\{\log[1 - F(x)]\} \,,$$

it then follows after integration that

$$F(x) = 1 - \exp[-\int_0^x h(y)dy] \; . \tag{2.3}$$

Thus if the hazard is specified, so is the distribution function, and conversely.

Note that if

$$h(x) = \lambda > 0 \; , \qquad 0 \leq x$$

then

$$F(x) = 1 - e^{-\lambda x} \; , \qquad 0 \leq x \; , \tag{2.4}$$

so a constant hazard function implies the exponential distribution of the random variable X, and conversely.

Obviously a constant hazard representation does not describe the bath tub hazard shape of Figure 1, nor does it represent a situation in which hazards decline, possibly because design defects or "bugs" are occasionally removed. Here are two hazard representations likely to be useful for such purposes.

1. A Bath Tub Model

Define the random variable Z in terms of X, X being exponentially distributed with mean λ^{-1}, as follows:

$$\begin{aligned} Z &= G(X) = XL(X)R(X) \\ \text{or} \quad &= X\phi(X), \quad \phi(X) = L(X)R(X) \end{aligned} \tag{2.5}$$

where

a) $L(x)$ is concave in x, $L(0) < 1$, $L(\infty) = 1$,

b) $R(x)$ is convex in x, $R(0) = 1$, $R(0) > R(\infty)$.

Then the hazard of Z may be made to exhibit a bath tub shape, as in Figure 1, by proper choice of the functions L and R.

Example. Suppose

$$L(x) = \frac{\alpha x}{1 + \alpha x} \qquad \alpha > 0,\ 0 \leq x \qquad\qquad (2.6)$$

$$R(x) = \frac{1}{1 + \beta x} \qquad \beta > 0\ ,$$

Clearly,

$$z = xL(x)R(x) = x \frac{\alpha x}{1 + \alpha x} \cdot \frac{1}{1 + \beta x} = \frac{\alpha x}{1 + \alpha x} \quad \frac{x}{1 + \beta x}$$

is a monotonically increasing function of x. Furthermore, choose α large (e.g. $\alpha = 10$) and β small (e.g. $\beta = 10^{-3}$). Then it is intuitively clear that (i) small x-values transform into even smaller z-values, e.g. $x = 1$ corresponds to $z = 0.91$ and $x = 2$ corresponds to $z = 1.90$, but (ii) this effect dwindles as x increases, so $x = 10$ corresponds to $z = 9.8$ and $x = 50$ to $z = 47.5$ and the z-values closely resemble the x's percentage-wise, but (iii) as x increases still further the z's do not follow suit: $x = 10^3$ corresponds to $z = 500$. This suggests that if x is a value assumed by X, that Z shares the properties of X in mid-range, i.e. for intermediate x-values, but differs from X by having a disproportionate probability of assuming small values (near zero), or large values (near, but less than, $1/\beta$). Thus the hazard of

Z will appear to be a "bath tubbed" version of X, particularly if X is exponential.

We focus attention on the representation (2.6) in what follows, mainly for analytical and computational convenience. Of course there are many other possibilities, such as

$$\begin{aligned} L(x) &= 1 - e^{-\alpha x} \\ R(x) &= e^{-\beta x} ; \end{aligned} \tag{2.7}$$

these latter may be adjusted to provide sharper-edged tubs than can (2.6), but iteration of (2.6) may be induced to accomplish the same purpose.

2. A Decreasing Failure Rate Model

Define the random variable W in terms of X, X again being exponential with parameter λ^{-1}:

$$W = XT(X) \tag{2.8}$$

where $T(x)$ is an increasing function of x, $L(0) = 1$. Then the hazard may be made to exhibit a decreasing behavior.

Example. Suppose

$$T(x) = 1 + cx , \qquad c > 0,\ 0 \leq x . \tag{2.9}$$

Then

$$z = x(1 + cx) \tag{2.10}$$

is monotonic, and small x-values lead to comparable z-values

(especially when c is small), but larger x-values are "amplified" by $1 + cx$ to yield increasingly large z-values.

Attention will be focused upon (2.9), although other possibilities exist that accomplish the same purpose, namely that of lengthening the right tail of the distribution of X (simulating outliers, for instance) while leaving the body of the distribution virtually unchanged.

B. MATHEMATICAL PROPERTIES OF THE "BATH TUB" HAZARD MODEL

Various analytical properties of the previously described models will now be recorded. These provide useful insights into the behavior of the random variables Z and the underlying (generating) variables X.

1. Monotonicity; Quantiles

It is convenient to focus on monotonic increasing transformations, i.e. if

$$z = G(z) = z\phi(z) \tag{2.10}$$

then in order that the above function be monotonically increasing, $dz/dx > 0$. Observe that logarithmic differentiation of (2.10) provides

$$\frac{dz}{z} = \frac{dx}{x} + \frac{\phi'(x)}{\phi(x)}\,dx \tag{2.11}$$

and thus $dz/dx > 0$ if and only if

$$\frac{1}{x} + \frac{\phi'(x)}{\phi(x)} > 0 \tag{2.12}$$

Alternatively, the condition is, in terms of $L(x)$ and $R(x)$,

$$\frac{1}{x} + \frac{L'(x)}{L(x)} + \frac{R'(x)}{R(x)} > 0 \tag{2.13}$$

It is easily seen that the important example (2.6),

$$\phi(x) = \frac{\alpha x}{1 + \alpha x} \cdot \frac{1}{1 + \beta x} ,$$

yields a monotonic relationship between z and x. The fact that this transformation can be easily and explicitly inverted (solved for x in terms of z) will be exploited subsequently.

Of course if $z(x)$ is monotonically increasing then so is $x(z)$, the inverse function. The events $(Z \leq z)$ and $(X \leq x(z))$ are equivalent, and so

$$P\{Z \leq z\} = P\{X \leq x(z)\} , \tag{2.14}$$

from which it follows that if $x_p \equiv x(p)$ is the $p \cdot 100\%$ quantile of X, i.e.

$$P\{X \leq x(p)\} = p , \tag{2.15}$$

then

$$P\{Z \leq z(p)\} = P\{Z \leq z(x(p))\} = p \tag{2.16}$$

and so $z(p)$, the $p \cdot 100\%$ quantile of Z is simply obtained from

$$z(p) = x(p)\ \phi(x(p)) = x(p)\ L(x(p))\ R(x(p)) \tag{2.17}$$

In other words we very easily translate from (points on)

the inverse distribution of X to the inverse distribution of Z. Explicit representation of the distribution of Z is however, not often easily possible.

2. Hazard and Density Function Relationships

In order to investigate the relationship between the hazards of Z and X, begin by writing

$$p = F_X(x(p)) = 1 - \exp[-\int_0^{x(p)} h_x(u)du] \tag{2.18}$$

or

$$\int_0^{x(p)} h_x(u)\,du = -\ln(1-p)$$

Now differentiate with respect to p to find

$$h_x(x(p)) \frac{dx(p)}{dp} = \frac{1}{1-p} \tag{2.19}$$

or

$$h_x(x(p)) = \frac{dp}{dx(p)} \cdot \frac{1}{1-p} = f_x(x(p)) \cdot \frac{1}{1-p} \; ; \tag{2.20}$$

here h_x and f_x are the hazard and density functions of the r.v. X. The relationship (2.20) holds for any distribution, of course.

Differentiation of (2.5) reveals the connection between h_z and h_x. From (2.11)

$$\frac{dz(p)}{dp} = z(p)\left[\frac{1}{x(p)} + \frac{\phi'(x(p))}{\phi(x(p))}\right]\frac{dx(p)}{dp} \tag{2.21}$$

$$= [\phi(x(p)) + x(p)\ \phi'(x(p))]\ \frac{dx(p)}{dp}$$

From (4.11), applied now to the z-hazard, there results

$$\frac{1}{h_z(z(p))} = \frac{1}{h_x(x(p))} [\phi(x(p)) + x(p)\ \phi'(x(p))] , \qquad (2.22)$$

so

$$h_z(z(p)) = h_x(x(p)) \frac{1}{\phi(x(p)) + x(p)\ \phi'(x(p))} \qquad (2.23)$$

Multiplication of both sides by $1-p$ then shows, in view of (4.11), that the density functions are similarly related:

$$f_z(z(p)) = f_x(x(p)) \frac{1}{\phi(x(p)) + x(p)\ \phi'(x(p))} .$$

Example.

X is exponential(λ). Then

$$h_z(x(p)) = \frac{\lambda}{\phi(x(p)) + x(p)\ \phi'(x(p))} \qquad (2.24)$$

Now use the specific $\phi(x)$ of (2.6):

$$\phi(x) = \frac{\alpha x}{1 + \alpha x} \cdot \frac{1}{1 + \beta x} ,$$

or, in terms of logarithms,

$$\ell n\ \phi(x) = \ell n\ \alpha x - \ell n(1 + \alpha x) - \ell n(1 + \beta x) ,$$

so

$$\frac{\phi'(x)}{\phi(x)} = \frac{1}{x} - \frac{\alpha}{1 + \alpha x} - \frac{\beta}{1 + \beta x} = \frac{1 - \alpha\beta x}{x(1 + \alpha x)(1 + \beta x)} , \qquad (2.25)$$

and

$$\phi(x) + x\phi'(x) = \phi(x) \left[\frac{2 + (\alpha + \beta)x}{(1 + \alpha x)(1 + \beta x)} \right] ; \qquad (2.26)$$

finally

$$h_z(z(p)) = \frac{\phi(1 + \alpha x(p))^2 \; (1 + \beta x(p))^2}{\alpha x(p) \; [2 + (\alpha + \beta) \; x(p)]} \qquad (2.27)$$

Although this expression is not quite explicit, qualitative properties of h_z can be deduced from it.

If $p \to 0$, $x(p) = -\frac{1}{\lambda} \ln(1-p) \downarrow 0$, and hence

$$h_z(z(p)) \sim \frac{1}{2\alpha \; x(p)}, \qquad (2.28)$$

or

$$\lim_{p \to 0} x(p) \; h_z(z(p)) = \frac{\lambda}{2\alpha} \qquad (2.29)$$

Since for $p \to 0$,

$$z(p) \sim \alpha x^2(p)$$

and hence

$$x(p) \sim [z(p)/\alpha]^{1/2} \qquad (2.30)$$

there results

$$h_z(z(p)) \sim \frac{\lambda}{2[z(p)]^{1/2} \sqrt{\alpha}}$$

or

$$\lim_{p \to 0} \sqrt{z(p)} \; h_z(z(p)) = \frac{\lambda}{2\sqrt{\alpha}} \qquad (2.31)$$

This shows that $h_z(z) \uparrow \infty$ as $z \to 0$, creating the left-hand end of the bath tub of Figure 1.

If $p \to 1$, $x(p) \uparrow \infty$, and $z(p) \uparrow 1/\beta$ so

$$h_z(z(p)) \sim \lambda \; \frac{\alpha\beta^2 \; x^2(p)}{(\alpha + \beta)}$$

or

$$\lim_{p \to \infty} \frac{1}{[x(p)]^2} h_z(z(p)) = \lambda \frac{\alpha\beta^2}{\alpha + \beta} \tag{2.32}$$

For $p \to 1$

$$1 - \beta z(p) \sim \frac{(\alpha + \beta)}{\alpha\beta \quad x(p)} \tag{2.33}$$

so

$$x(p) \sim \frac{\alpha + \beta}{\alpha\beta} \frac{1}{1 - \beta z(p)} \tag{2.34}$$

and thus

$$h_z(z(p)) \sim \lambda\left(\frac{\alpha + \beta}{\alpha}\right) \frac{1}{(1 - \beta z)^2} \tag{2.35}$$

Once again it appears that the hazard rises rapidly, this time as $x(p) \uparrow \infty$ and $z(p) \uparrow \beta^{-1}$; the other end of the bath tub is thus fashioned.

If $p = 1 - e^{-1}$, then $x(p) = \lambda^{-1}$. Then

$$h_z(z(1 - e^{-1})) = \frac{[1 + \alpha/\lambda]^2 \; [1 + \beta/\lambda]^2}{\alpha/\lambda \; [2 + (\alpha + \beta)/\lambda]} \tag{2.36}$$

The bath tub effect is presumably achieved by choosing α large and β small. Let $\alpha \to 0$ and $\beta \to 0$ independently in (2.36); it is clear that the limiting value of the hazard is λ. This indicates that the hazard is (approximately) λ for middling values of z.

3. An Explicit Formula for a Hazard

The expression (2.6) leads to the relationship

$$z(p) = \frac{\alpha x^2(p)}{(1 + \alpha x(p))\,(1 + \beta x(p))}, \tag{2.37}$$

and the latter may be explicitly inverted by solving a quadratic equation. The result is

$$x(p) = \frac{(\alpha+\beta)z(p) + \sqrt{(\alpha+\beta)^2 z^2(p) + 4z(p)\,\alpha(1-\beta z(p))}}{2\alpha(1-\beta z(p))} \tag{2.38}$$

Now a direct differentiation of this expression and invocation of (2.20) produces the expression

$$h_z(z)$$

$$= \frac{\lambda}{2\alpha(1-\beta z)}\left[\frac{\beta\{(\alpha+\beta)z + \sqrt{(\alpha+\beta)^2 z^2 + 4\alpha z(1-\beta z)}}{1-\beta z} + (\alpha+\beta)\right.$$

$$\left. + \frac{\{(\alpha-\beta)^2 z + 2\alpha\}\sqrt{(\alpha+\beta)^2 z^2 + 4\alpha z(1-\beta z)}}{(\alpha+\beta)^2 z^2 + 4\alpha z(1-\beta z)}\right] \tag{2.39}$$

This form, while explicit, provides no particularly useful insights; the bath tub end shapes already noted in (2.31) and (2.35) can be deduced directly from (2.39).

Some graphical plots of h_z are presented below. They illustrate the behavior of the present hazard representation in a more understandable fashion than does the formula itself.

4. An Explicit Formula for the Failure Time Distribution

Because x and z are monotonically related through (2.37) we have

$$P\{Z \leq z\}$$

$$= P\left\{X \leq x = \frac{(\alpha+\beta)z + \sqrt{(\alpha+\beta)^2 z^2 + 4\alpha z(1-\beta z)}}{2\alpha(1 - \beta z)}\right\}$$

$$= 1 - \exp\left\{-\lambda\left[\frac{(\alpha+\beta)z + \sqrt{(\alpha+\beta)^2 z^2 + 4\alpha z(1 - \beta z)}}{2\alpha(1 - \beta z)}\right]\right\} \tag{2.40}$$

Again the explicit formula seems unproductive of insights.

C. MATHEMATICAL PROPERTIES OF THE DECREASING FAILURE RATE MODEL

1. The Hazard Behavior

The expression (2.23) can be applied to deduce the hazard function of the representation (2.8), advertised to produce a decreasing failure rate. There we specified

$$\phi(x) \equiv T(x) = 1 + cx , \tag{2.41}$$

and thus, from (2.23)

$$h_W(w(p)) = \frac{\lambda}{(1 + cx(p)) + x(p)c} = \frac{\lambda}{1 + 2cx(p)} \tag{2.42}$$

Qualitative properties follow easily.

If $p \to 0$, $x(p) \downarrow 0$, and

$$h_W(w(p)) \sim \lambda \quad . \tag{2.43}$$

Thus the hazard is approximately λ for small z.

If $p \to 1$, $x(p) \uparrow \infty$, and

$$w(p) \sim c[x(p)]^2$$

so

$$h_w(w(p)) \sim \frac{\lambda}{2\sqrt{cw(p)}} \, , \tag{2.45}$$

which clearly decreases, as claimed. It may be inferred that the distribution of W appears nearly exponential, but possesses an extraordinarily long right tail--possibly the result of outliers.

2. Explicit Formulas for the Hazard and the Distribution Function

Direct solution of the quadratic equation

$$w = x(1 + cx) = x + cx^2$$

presents

$$x(p) = \frac{\sqrt{1 + 4cw(p)} - 1}{2c} \, , \tag{2.46}$$

which, when differentiated, leads to

$$\begin{aligned} h_w(w) &= \frac{1}{\sqrt{1 + 4cw}} h_x(x) \\ &= \frac{\lambda}{\sqrt{1 + 4cw}} \end{aligned} \tag{2.47}$$

The distribution function is

$$\begin{aligned} P\{W \leq w\} &= P\left\{ X \leq x = \frac{\sqrt{1 + 4cw} - 1}{2c} \right\} \\ &= 1 - \exp\left\{ -\lambda \left[\frac{\sqrt{1 + 4cw} - 1}{2c} \right] \right\} \end{aligned} \tag{2.48}$$

This distribution bears a close family resemblance to the Weibull distribution $1 - F(w) = \exp\{-k\sqrt{w}\}$, especially for large (right tail) values of w.

D. AN ALTERNATIVE "BATHTUB" HAZARD REPRESENTATION

The simple parametric model (2.6) leading to a bathtub-shaped hazard is by no means the only possibility. We next describe another simple approach. It is that of defining a hazard function having an appropriate shape, and then deducing the corresponding distribution function, and a procedure for sampling from it, rather than proceeding in reverse order, as before.

Let the hazard be of the form

$$h(z) = g(z) + \cdots + k(z) , \tag{2.49}$$

where $g(z) > 0$ is a decreasing function of z such that $\lim_{z \to \infty} g(z) = 0$, and $k(z)$ is an increasing function of z, such that (preferably) $k(0) = 0$ and $k(\infty) = \infty$. Such a function can yield a bathtubbed hazard.

Example.

$$h(z) = \frac{A}{z + \alpha} + Bz + \lambda \tag{2.50}$$

A, B, α, λ all positive.

Clearly, (2.50) has a generally "bath tub-like" appearance, since

$$h'(z) = -\frac{A}{(z + \alpha)^2} + B \tag{2.51}$$

if

$$-\frac{A}{\alpha} + B < 0, \quad \text{then} \quad h'(0) < 0 \tag{2.52}$$

while for

$$z > z_0 = \frac{A}{B} - \alpha \tag{2.53}$$

$h(z) > 0$.

Detailed behavior is adjustable by choice of the parameters. Now the distribution function of time to failure, Z, is obtained from (2.50):

$$
\begin{aligned}
P\{Z > z\} &= \exp\left\{- \int_0^z h(x)\,dx\right\} \\
&= \exp\left\{- \int_0^z \left(\frac{A}{x + \alpha} + Bx + \lambda\right) dx\right\} \\
&= \exp\left\{- \left[A \,\ln\left(1 + \frac{z}{\alpha}\right) + \frac{B}{2} z^2 + \lambda z\right]\right\} \\
&= \left(\frac{\alpha}{\alpha + z}\right)^A \exp\left(- \frac{B}{2} z^2\right) e^{-\lambda z} \\
&= \bar{F}_1(z)\, \bar{F}_2(z)\, \bar{F}_3(z) ,
\end{aligned}
\tag{2.54}
$$

where

$$
\begin{aligned}
\bar{F}_1(z) &= \left(\frac{\alpha}{\alpha + z}\right)^A \\
\bar{F}_2(z) &= \exp\left(- \frac{B}{2} z^2\right) \\
\bar{F}_3(z) &= e^{-\lambda z} .
\end{aligned}
\tag{2.55}
$$

All of the above are recognized as being the complements of distribution functions. In effect, the distribution of Z is that of the minimum of three independent random variables:

$$P\{Z > z\} = P\{X_1 > z\} \cdot P\{X_2 > z\} \cdot P\{X_3 > z\}, \tag{2.56}$$

X_i having the distribution $F_i = 1 - \bar{F}_i$ $(i = 1,2,3)$. This fact leads directly to an easy procedure for simulation of Z by simply obtaining the smallest from among the realization of X_1, X_2, and X_3. The advantage of the previous method, based on (2.6) for instance, is that only one realization--that of an exponential in that specific example--leads to the realization of Z. This is not only computationally attractive, but seems to facilitate the application of such Monte Carlo variance reduction techniques as control and antithetic variables, cf., Hammersley and Handscomb [5].

III. OBTAINING SPECIFIED HAZARD BEHAVIOR BY SIMPLE SAMPLING

The development of the last section illustrates one manner in which hazard behavior may be conveniently represented and simulated. We now show how such behavior may alternatively be obtained by simple simulation, i.e. from one realization of a basic (possibly exponential) random variable.

Refer to (2.5), in which

$$Z = G(X) \tag{3.1}$$

and, if $G(\cdot)$ is monotonically increasing,

$$z(p) = G(x(p)) \ , \tag{3.2}$$

$z(p)$ and $x(p)$ being the $p \cdot 100\%$ percentiles of Z and X, respectively. Then the counterpart to (2.23) that results from differentiation of (3.2) is the expression

$$h_z(z(p)) = h_x(x(p)) \frac{1}{G'(x(p))} = h_x(x(p)) \frac{1}{(dz/dx)} \tag{3.3}$$

Consequently, if one specifies $h_z(z)$ as a suitable function of the "time" z, and specifies the distribution of the stochastic variable X--and hence its hazard, h_x--there results a differential equation for $z(x) \equiv G(x)$:

$$h_z(z) \frac{dz}{dx} = h_x(x) \ ; \tag{3.4}$$

integration then provides the desired transformation, G. In other words, we seek $z(x)$ satisfying

$$\int_0^z h_z(u)\,du = \int_0^x h_x(v)\,dv\ , \tag{3.5}$$

which can sometimes be carried out in a useful closed form.

Example 3.1. Refer to the example of Section II, wherein h_z is given by the expression (2.50) and we assume that X is exponential, so h_x is constant. Then in order to determine $G(x) \equiv z(x)$, solve the equation

$$\int_0^z \left[\frac{A}{u+\alpha} + Bu + \lambda\right] du = \int_0^x dv$$

or

$$A\,\ell n(1 + \frac{z}{\alpha}) + \frac{B}{2} z^2 + \lambda z = x \tag{3.6}$$

Closed-form solution of this expression for z in terms of x is of course impossible. One possible approach is purely numerical: find an approximate solution, $z_0(x)$, e.g. the appropriate solution of the quadratic

$$\frac{B}{2} z^2 + \lambda z = x \tag{3.7}$$

and then correct the result by a few Newton-Raphson iterations. In other words, put

$$z_1(x) = \frac{-\lambda + \sqrt{\lambda^2 + 2Bx}}{B}\ ; \tag{3.8}$$

now apply Newton to obtain an improved solution

$$z_2(x) = z_1 - \frac{A\,\ell n(1 + z_1/\alpha)}{A/(z_1 + \alpha) + Bz_1 + \lambda} \tag{3.9}$$

which will be feasible if $0 < z_2$. The process can be iterated (the numerator will change after the first iteration). If one wishes to use this model it may actually be desirable to start by solving

$$\frac{B}{2} z^2 + (\lambda + \frac{A}{\alpha}) z - x = 0 \qquad (3.10)$$

for z_1, in which case the numerator will not be as shown in (3.9); convergence may be more rapid.

Example 3.2. Change the hazard representation of the previous example as follows: let

$$h_z(u) = \frac{A}{(v + \alpha)^2} + Bu + \lambda; \qquad (A, \alpha, B, \lambda > 0) \qquad (3.11)$$

then

$$\int_0^z h_z(u) du = \frac{Az}{\alpha(z + \alpha)} + \frac{B}{2} z^2 + \lambda z \qquad (3.12)$$

Now it is necessary to solve

$$\frac{Az}{\alpha(z + \alpha)} + \frac{B}{2} z^2 + \lambda z = x , \qquad (3.13)$$

i.e., the cubic

$$\frac{B}{2} z^3 + [\alpha \frac{B}{2} + \lambda] z^2 + [\frac{A}{\alpha} + \alpha\lambda - x] z - \alpha x = 0 , \qquad (3.14)$$

which can be carried out, at least formally, in closed form. Once again an iterative solution that begins by dropping the cubic term, solving the resulting quadratic for $z_1(x)$, and then continuing along the Newton-Raphson road may be successful. Further investigations of these ideas should be conducted.

IV. COMPUTER SIMULATION AND ESTIMATION PROCEDURE

A. SIMULATION AND NUMERICAL RESULTS

In previous chapters an analytical model was described for the failure rate function; useful formulas were also derived from the model (2.39) and (2.40). Before the model is used in realistic situations, it will be convenient to build a computer simulation model for model validation.

1. Algorithm

First a very basic simulation model was built for determining the general shape of the failure rate function associated with parameters α, β and λ. In the simulation model, α was selected to be 1.0 and 2.0, β was selected to be 0.05, 0.01, 0.005, 0.001 and λ was selected to be 0.1. These values were picked arbitrarily, it is only stipulated that α is always greater than β. The general algorithm of the simulation model is shown in Fig. 2.

The hazard function is calculated according to the model (2.37) and the system logic function (2.40). The results were shown in Fig. 3 and Fig. 4.

2. Some Comments

These simulation results show that:

a. Parameter α is effective when z values relatively have small values. That is, it influences the early failure period.

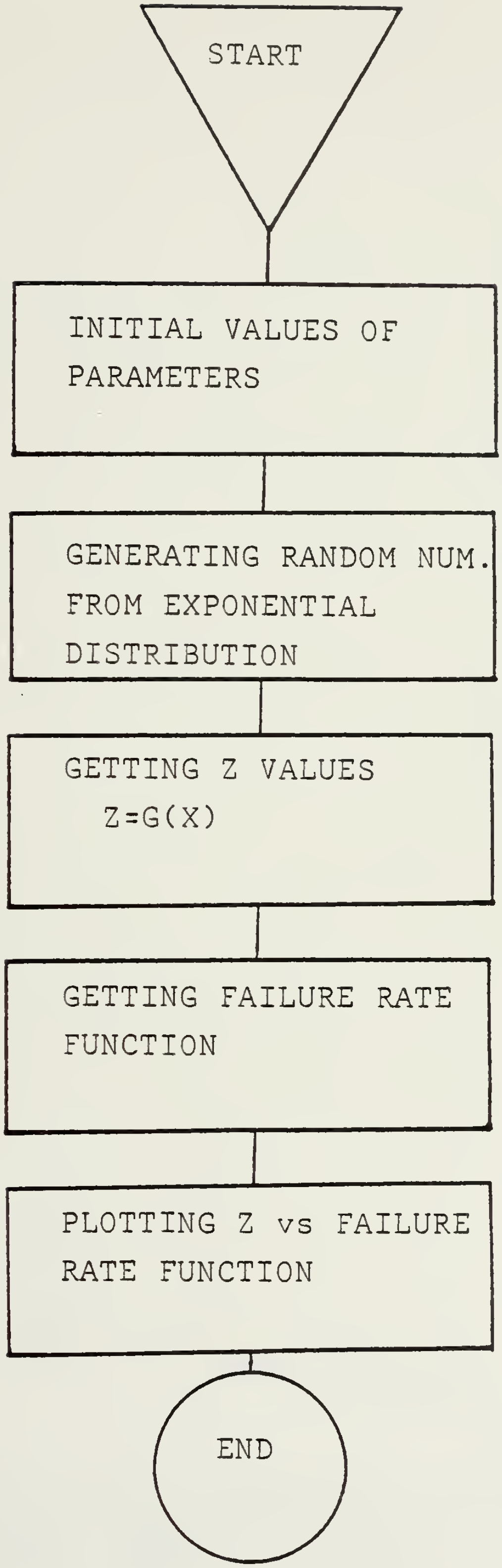
START
INITIAL VALUES OF PARAMETERS
GENERATING RANDOM NUM. FROM EXPONENTIAL DISTRIBUTION
GETTING Z VALUES Z=G(X)
GETTING FAILURE RATE FUNCTION
PLOTTING Z vs FAILURE RATE FUNCTION
END

Fig. 2. Basic Algorithm.

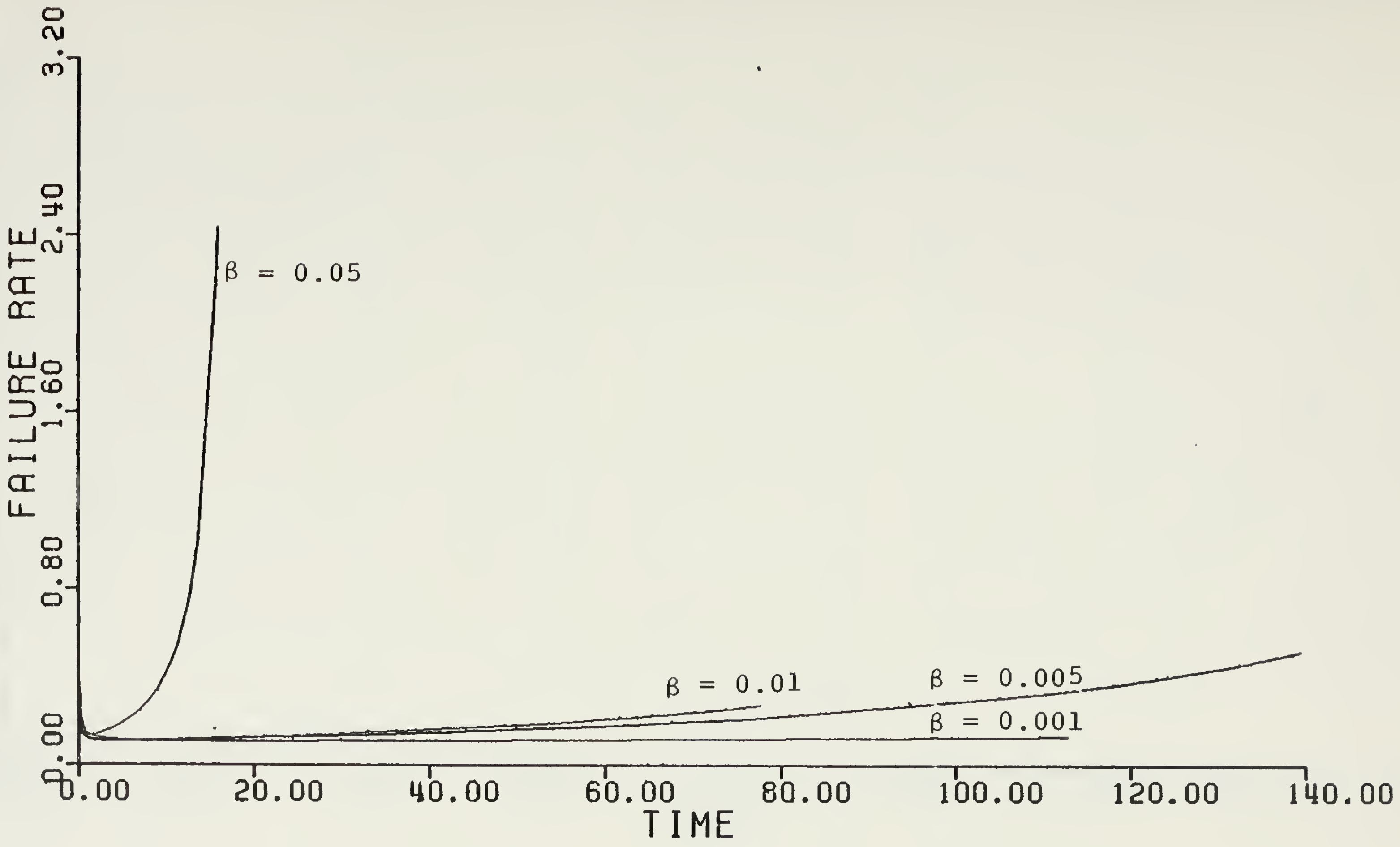

Fig. 3. The Shape of Hazard Function with Different Values β $\alpha = 1.0$, $\lambda = 0.1$.

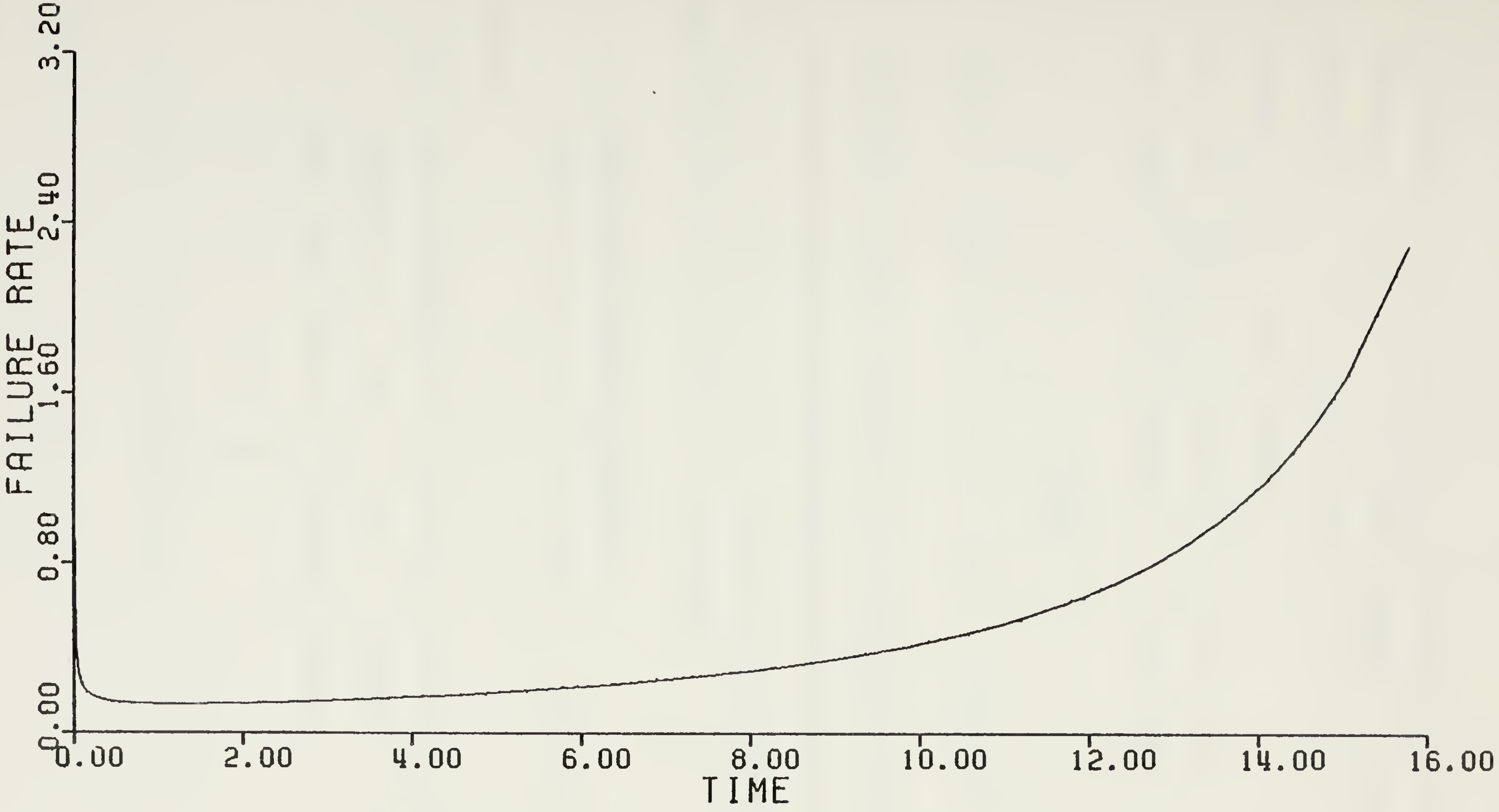

Fig. 4. The Hazard Function Over Its Range

b. Parameter β is effective on the relatively bigger z values. That is, it describes wearout failures.

c. Parameter λ has little effect on the shape of the curve; it is a scale factor.

d. The last important observation is that the z values are limited by the parameter β such that:

$$z \leq \frac{1}{\beta}$$

When z equals $1/\beta$, $h_z(z)$ goes to infinity.

B. EMPIRICAL DENSITY FUNCTION AND ESTIMATION PROCEDURE

In this section, an estimation procedure for parameters α, β, λ is defined and the procedure of finding the empirical density function is described.

1. Empirical Density Function

Suppose that N_i, N, Δ and N_0 are defined as follows:

N_i = the frequency of data points for each time interval between a_i and b_i for i equals 1,2,3,...,k.

N = the total number of failures where,

$$N = \sum_{i=1}^{k} N_i$$

Δ = the length of each interval

$$\Delta = b_i - a_i$$

N_0 = the total number of survivors by the beginning of each interval which can be shown as follows:

$$N_0 = \begin{cases} N & \text{if } i = 1 \\ N - \sum_{j=1}^{i-1} N_j & \text{if } i = 2,3,\ldots,k \end{cases}$$

where $\sum_{j=1}^{i=1} N_j$ is the number of failures before interval i, or $N - \sum_{j=1}^{i=1} N_j$ is the number of failures after interval i.

If $h_z(z)$ is defined as the density function of hazard, then the relationship between the frequences of the failure data and density function of hazard can be given as follows:

$$N_i \approx N_0 \cdot \int_{a_i}^{b_i} h_z(z)\,dz \tag{4.1}$$

where $i = 1,2,3,\ldots,k$ and $a_i = (i-1)\Delta$, $b_i = i\Delta$. At this point some approximation can be made in expression (4.1), such that

$$\int_{a_i}^{b_i} h_z(z)\,dz \approx (b_i - a_i)\cdot h_z(i(b_i - a_i)) \tag{4.2}$$

for small values of $\Delta = b-a$. Then this approximation (4.2) is substituted in the expression (4.1), it turns out as follows:

$$N_i = \Delta \cdot N_0\, h_z(i\Delta) \tag{4.3}$$

or

$$\hat{h}_z(i\Delta) = \frac{N_i}{N_0 \cdot \Delta} , \qquad i = 1,2,\ldots,k \tag{4.4}$$

for the general case, it will be:

$$\hat{h}_z(i\Delta) = \begin{cases} \dfrac{N_i}{N} & \text{for} \quad i = 1 \\ \dfrac{N_i}{(N - \sum\limits_{j=1}^{i-1} N_j)} \dfrac{1}{\cdot\Delta} & \text{for} \quad i = 2,3,\ldots \end{cases} \tag{4.5}$$

The expression (4.5) will be used in the calculation of an empirical hazard function and also used in the proposed estimation procedure.

2. Estimation Procedure for Model Parameters

Two approaches can be used for this problem. The first idea is to use the relationship between a sample pth percentile and the related probability of the pth percentile. A subsequent idea is to approach the problem as a nonlinear least square estimation problem for α, β, λ: pick α, β λ so that the values obtained minimize the sum of squared errors in an objective function.

a. 3-Percentile Approach

Suppose $F_X(x)$ is the cumulative probability function of the exponential distribution. The pth percentile $x(p)$ is equal to the value such that:

$$p = F_X(x(p)) = 1 - e^{-\lambda x(p)} \tag{4.6}$$

Then

$$x(p) = F^{-1}(p) = -\frac{1}{\lambda}\ \ell n(1-p) \tag{4.7}$$

where $\lambda > 0$.

If $z(p)$ is defined to be the pth percentile in the bathtub model (2.37) then $z(p)$ and $x(p)$ have a definite relationship with expression such that:

$$z(p) = \frac{\alpha x^2(p)}{(1 + \alpha x(p))\ (1 + \beta x(p))}$$

by substituting (4.7),

$$z(p) = \frac{\alpha\{-\frac{1}{\lambda}\ \ell n(1-p)\}^2}{\left\{1 + \alpha\left[-\frac{1}{\lambda}\ \ell n(1-p)\right]\right\}\left\{1 + \beta\left[-\frac{1}{\lambda}\ \ell n(1-p)\right]\right\}} \tag{4.8}$$

or

$$z(p) = \frac{(\alpha/\lambda^2)\,\varepsilon^2(p)}{\left\{1 + \frac{\alpha}{\lambda}\ \varepsilon(p)\right\}\left\{1 + \frac{\beta}{\lambda}\ \varepsilon(p)\right\}} \tag{4.9}$$

where $\varepsilon(p) = -\ \ell n(1-p)$.

In the last equation (4.9), there are three unknown variables which are α, β, λ. If three independent equations associated with expression (4.9) are defined, clearly, they can be solved for the unknown parameters. Actually, this can be easily done, for using different values of percentiles such that:

$$z(P_1) = \frac{(\alpha/\lambda^2)\varepsilon^2(P_1)}{\left\{1 + \frac{\alpha}{\lambda}\varepsilon(P_1)\right\}\left\{1 + \frac{\beta}{\lambda}\varepsilon(P_1)\right\}}$$

$$z(P_2) = \frac{(\alpha/\lambda^2)\varepsilon^2(P_2)}{\left\{1 + \frac{\alpha}{\lambda}\varepsilon(P_2)\right\}\left\{1 + \frac{\beta}{\lambda}\varepsilon(P_2)\right\}}$$

$$z(P_3) = \frac{(\alpha/\lambda^2)\varepsilon^2(P_3)}{\left\{1 + \frac{\alpha}{\lambda}\varepsilon(P_3)\right\}\left\{1 + \frac{\beta}{\lambda}\varepsilon(P_3)\right\}}$$

where $z(P_i)$ and $\varepsilon(P_i)$ are the known values associated with P_i. Actually, to get the percentile values, two kinds of approaches can be made. First they can be computed directly from data using the simple statistics method, such as $z_{(i)}$ is (approximately) $z(i/(n+1))$. Second they can be computed from the empirical density function. Generally the choice depends on the form of the data that are available.

To decide for the effectiveness of this type of estimation, another basic computer simulation is made by modifying the first computer simulation algorithm. This algorithm is shown in Fig. 5.

The simulation was run four hundred times and in each run the sample size was assumed to be $n = 50$. Initially the parameters α, β and λ were taken to be, respectively, 1.0, 0.05 and 0.1. The percentiles P_1, P_2 and P_3 were used in each run such that 0.1, 0.5 and 0.9 respectively. The estimation results are shown in Table I.

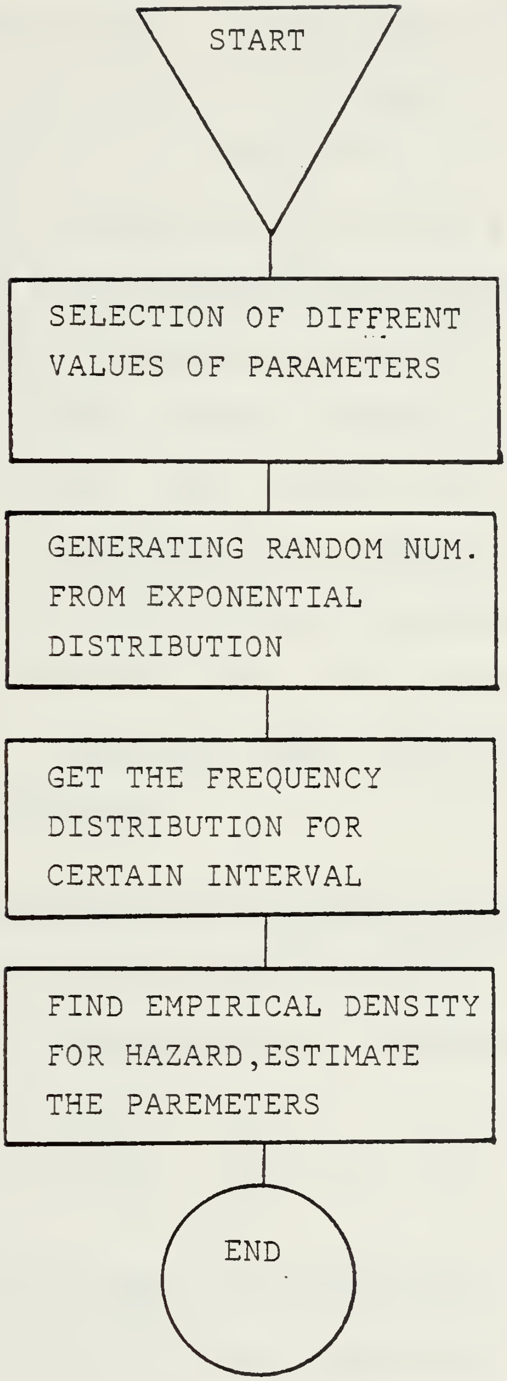
START
SELECTION OF DIFFRENT VALUES OF PARAMETERS
GENERATING RANDOM NUM. FROM EXPONENTIAL DISTRIBUTION
GET THE FREQUENCY DISTRIBUTION FOR CERTAIN INTERVAL
FIND EMPIRICAL DENSITY FOR HAZARD,ESTIMATE THE PAREMETERS
END

Fig.5.Simulation algoritm for estimation procedure and emprical density function.

As can be seen, the estimated parameters α, β, λ have quite different values in all runs. To judge their sampling distributions, histograms were drawn separately; Figures 6, 7 and 8 are the histograms of the estimated values of α, β, λ, respectively.

It is suggested by Figures 6 and 7 for λ and β, that normality may be assumed with sample means 0.9246, 0.05488 and sample variances 0.001065, 0.0002246, respectively. Their mean, median, trimean and midmean are pretty close to each other. But, in the case of α, the histogram (Fig. 8) is a quite different picture which looks something like an exponential distribution instead of normal distribution. The reason is clear, because the negative values of estimated α were not taken into account. They are physically infeasible.

b. Nonlinear Least Squares Approach

The least squares criterion used here can be stated formally and generally as follows:

$$\text{minimize} \quad \sum_{i=1}^{N} (Y_i - \hat{Y}_i)^2 \tag{4.10}$$

where N is the number of observations, Y_i is the fitted value of Y_i. In this case, expression (4.10) can be rewritten as follows

$$\min \sum_{i=1}^{N} \{Z_i - \hat{Z}_i(\alpha,\beta,\lambda)\}^2 \tag{4.11}$$

TABLE I

COMPUTER SIMULATION RESULTS FOR ESTIMATION PROCEDURE

Sample size $n = 50$; Number of Total Runs $m = 400$;

Value of parameters α, β, λ Introduced: $\alpha=1$, $\beta=0.05$, $\lambda=0.1$.

Number of Runs	λ	α	β
1	0.0858	2.2255	0.0549
2	0.06591	1.2745	0.0779
3	0.08381	9.498	0.0688
4	0.0881	0.9304	0.039
5	0.1286	3.7429	0.0498
6	0.0883	2.114	0.069
7	0.0785	2.8281	0.0593
8	0.0945	0.4073	0.0584
9	0.089	0.3833	0.0452
10	0.1344	1.3351	0.0426
11	0.0677	0.2111	0.0729
12	0.1304	2.4871	0.0512
13	0.0714	0.3211	0.0623
14	0.069	0.3547	0.0657
15	0.0775	0.4813	0.0772
16	0.1618	2.7416	0.027
17	0.0827	0.927	0.0705
18	0.0816	1.1663	0.0836
19	0.0515	0.3777	0.0775
20	0.1089	1.6439	0.0522
21	0.0767	0.8667	0.0605
22	0.0499	0.3414	0.0748
23	0.0569	0.2118	0.0608
24	0.0648	0.2689	0.0715
25	0.1415	2.1178	0.0518
26	0.1636	1.3635	0.0186
27	0.0947	6.101	0.0664

TABLE I Cont.

Number of Runs	λ	α	β
28	0.1063	0.5235	0.0623
29	0.0825	2.6602	0.059
30	0.13	0.3412	0.0303
31	0.817	0.24	0.0685
32	0.0843	0.4377	0.0393
33	0.1445	4.6941	0.0442
34	0.0737	0.2956	0.0665
35	0.045	0.1372	0.0646
36	0.1108	1.1026	0.0427
37	0.1277	1.091	0.052
38	0.0514	0.1362	0.062
39	0.1165	0.6794	0.0469
40	0.0954	3.7845	0.0587
41	0.0563	0.4123	0.0738
42	0.0684	0.2903	0.0574
43	0.1263	2.059	0.0357
44	0.1335	7.1595	0.0226
45	0.0708	0.465	0.063
46	0.0872	2.164	0.0657
47	0.0834	0.3568	0.0567
48	0.0429	0.127	0.0694
49	0.0428	0.1295	0.0639
50	0.0834	0.6807	0.0489
51	0.1141	1.5349	0.0566
52	0.1057	1.1757	0.0441
53	0.1104	5.0692	0.0433
54	0.1426	0.8845	0.0244
55	0.1084	0.5775	0.0505
56	0.1048	1.8399	0.0562
57	0.0561	0.3197	0.0643
58	0.0607	0.4564	0.0607
59	0.0724	0.3133	0.0551

TABLE I Cont.

Number of Runs	λ	α	β
60	0.0957	0.9584	0.0557
61	0.0804	0.7378	0.0423
62	0.0888	0.1718	0.0393
63	0.064	0.3779	0.9543
64	0.0324	0.2202	0.0791
65	0.0695	0.2428	0.0564
66	0.0903	6.4194	0.0542
67	0.0741	0.4118	0.0718
68	0.0462	0.167	0.0671
69	0.0805	1.0279	0.0662
70	0.0469	0.2157	0.0693
71	0.0487	0.2457	0.0688
72	0.0682	1.1847	0.0665
73	0.0538	0.1964	0.0573
74	0.1516	2.099	0.0352
75	0.0763	0.3067	0.0679
76	0.0558	1.5589	0.079
77	0.0463	0.2252	0.0612
78	0.095	5.6243	0.0627
79	0.1903	6.0	0.0604
80	0.0822	0.9989	0.0738
81	0.066	0.3422	0.0926
82	0.0784	0.3336	0.0517
83	0.0499	0.1453	0.062
84	0.0667	0.1357	0.064
85	0.0431	0.0761	0.0568
86	0.1485	6.1674	0.0262
87	0.1154	1.0451	0.0476
88	0.0043	0.3493	0 0648
89	0.0983	1.3101	0.0422
90	0.138	6.7408	0.0443

TABLE I Cont.

Number of Runs	λ	α	β
91	0.1117	0.7281	0.058
92	0.065	0.4284	0.0482
93	0.0638	1.2754	0.0611
94	0.1603	1.003	0.0302
95	0.0436	0.1647	0.085
96	0.0905	6.2455	0.0703
97	0.1309	1.4348	0.0431
98	0.0787	0.5245	0.0686
99	0.1927	7.801	0.009
100	0.0683	0.525	0.0698
101	0.1163	1.7146	0.0355
102	0.1005	1.5795	0.0463
103	0.1492	1.5169	0.0465
104	0.1511	2.7243	0.318
105	0.0823	0.1634	0.0438
106	0.1493	6.9467	0.0437
107	0.1185	9.4492	0.0603
108	0.1039	0.589	0.0406
109	0.0832	0.3935	0.0612
110	0.059	0.145	0.0508
111	0.133	8.6071	0.0397
112	0.0877	0.4751	0.0448
113	0.0772	0.2752	0.0624
114	0.0659	0.4782	0.0583
115	0.0796	0.2978	0.0537
116	0.1128	1.668	0.0601
117	0.0974	1.2931	0.0515
118	0.1105	0.3394	0.0468
119	0.1216	1.0529	0.0655
120	0.1046	2.0384	0.0591
121	0.128	3.2468	0.0405

TABLE I Cont.

Number of Runs	λ	α	β
122	0.0818	1.092	0.0584
123	0.0924	2.4295	0.0439
124	0.0732	0.1916	0.0495
125	0.1284	1.8515	0.0407
126	0.1376	2.6651	0.0282
127	0.0877	7.8429	0.0703
128	0.0865	1.1942	0.0438
129	0.1404	2.8991	0.0326
130	0.0452	0.0978	0.0663
131	0.0928	3.0463	0.0471
132	0.0668	0.4602	0.0551
133	0.2089	0.7373	0.0119
134	0.0946	0.4011	0.038
135	0.1352	1.2162	0.0284
136	0.0597	0.2241	0.0539
137	0.106	1.7821	0.0354
138	0.0508	0.2689	0.0596
139	0.0522	0.1244	0.055
140	0.104	0.8464	0.0603
141	0.1002	0.817	0.0711
142	0.0692	1.4214	0.0744
143	0.0995	0.5442	0.0435
144	0.0697	0.4966	0.0487
145	0.0737	0.8995	0.0708
146	0.059	0.2998	0.0654
147	0.1235	1.1251	0.045
148	0.1023	0.615	0.0349
149	0.1	0.9087	0.0466
150	0.1242	2.9267	0.0356
151	0.0966	0.4989	0.0616
152	0.0913	2.5416	0.0638
153	0.1301	1.3897	0.0434

TABLE I Cont.

Number of Runs	λ	α	β
154	0.1033	2.0535	0.0466
155	0.0638	0.2219	0.0594
156	0.1408	3.6331	0.0434
157	0.1311	4.9526	0.0495
158	0.0608	0.469	0.0462
159	0.0574	0.5886	0.0723
160	0.0998	4.912	0.0554
161	0.1114	0.4898	0.0363
162	0.0415	0.1827	0.0854
163	0.1586	7.5257	0.0397
164	0.0723	0.9055	0.0637
165	0.0984	2.1521	0.0675
166	0.1011	2.1535	0.0573
167	0.093	4.8488	0.0555
168	0.1141	4.2636	0.0443
169	0.1953	4.6512	0.0452
170	0.0568	0.8364	0.07099
171	0.0927	5.1303	0.068
172	0.0632	0.7399	0.069
173	0.0536	0.2732	0.07138
174	0.1033	0.9515	0.0787
175	0.0633	1.6597	0.0791
176	0.1414	1.2547	0.0318
177	0.1089	4.1786	0.051
178	0.1165	0.5441	0.0432
179	0.1731	4.2913	0.0192
180	0.0679	0.1064	0.0451
181	0.1682	3.1417	0.0332
182	0.0868	0.6399	0.0685
183	0.0807	1.046	0.0607
184	0.0778	1.185	0.0694

TABLE I Cont.

Number of Runs	λ	α	β
185	0.0769	0.1999	0.0009
186	0.0396	0.1194	0.064
187	0.1012	0.499	0.485
188	0.1389	9.7085	0.0334
189	0.0665	1.1661	0.0599
190	0.0471	0.1374	0.0643
191	0.1316	4.1676	0.0259
192	0.0687	0.6692	0.0928
193	0.0866	0.5419	0.0484
194	0.0966	1.579	0.0524
195	0.0773	1.0102	0.0668
196	0.1076	0.7413	0.0704
197	0.0857	0.8715	0.0548
198	0.0803	1.110	0.0549
199	0.052	0.1396	0.0598
200	0.0757	0.9558	0.0622

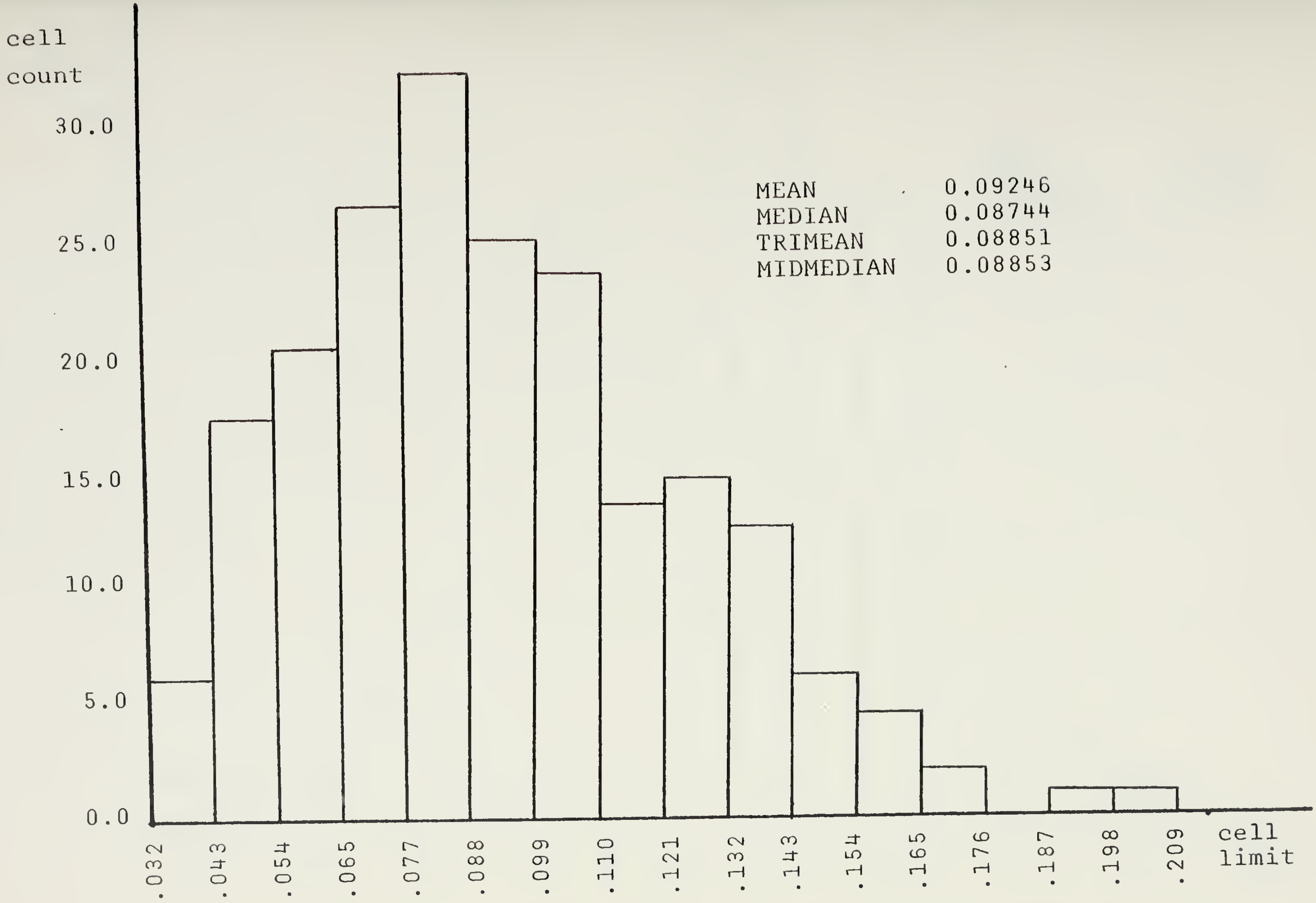

FIG. 6. Histogram for λ

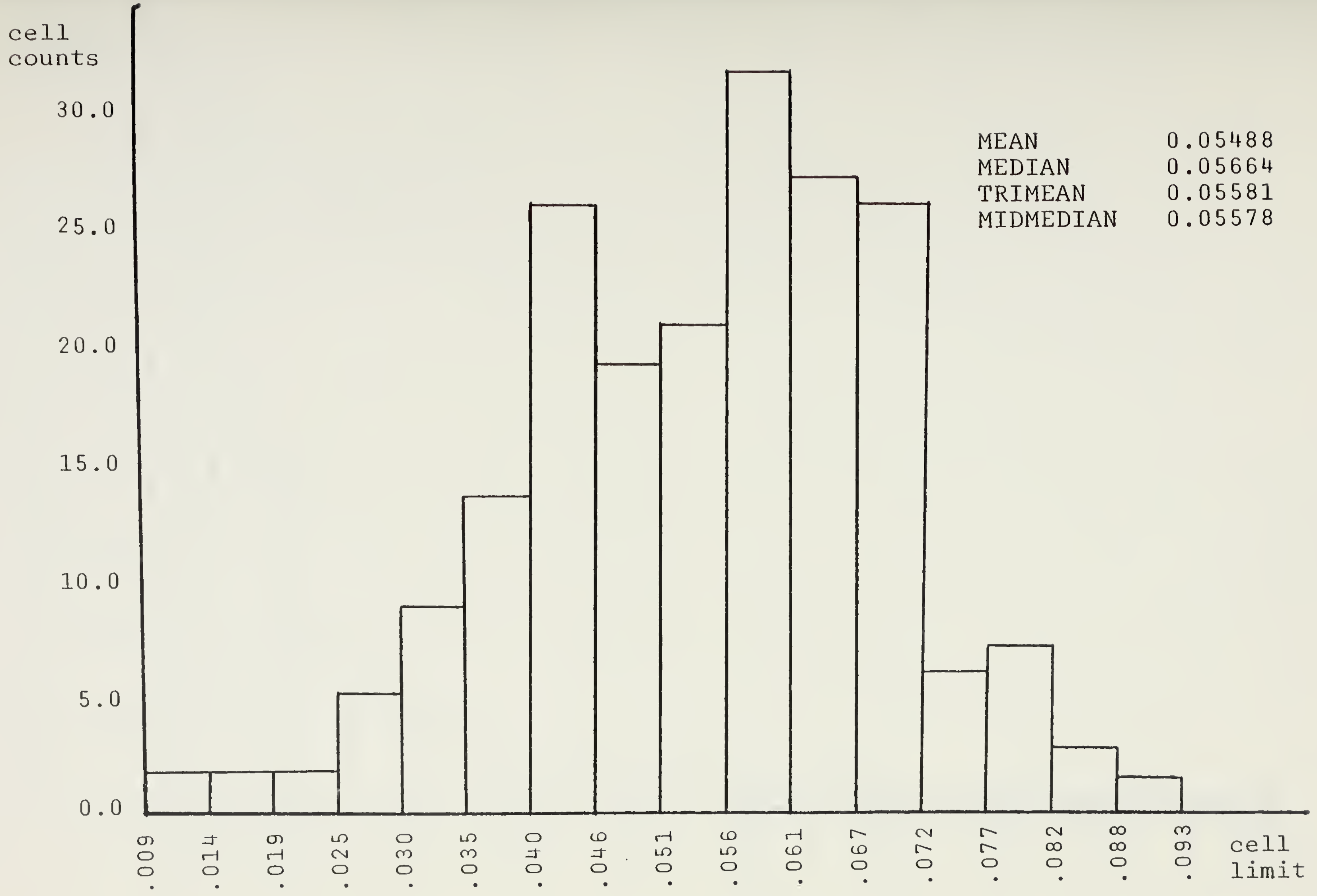

FIG. 7. Histogram for β

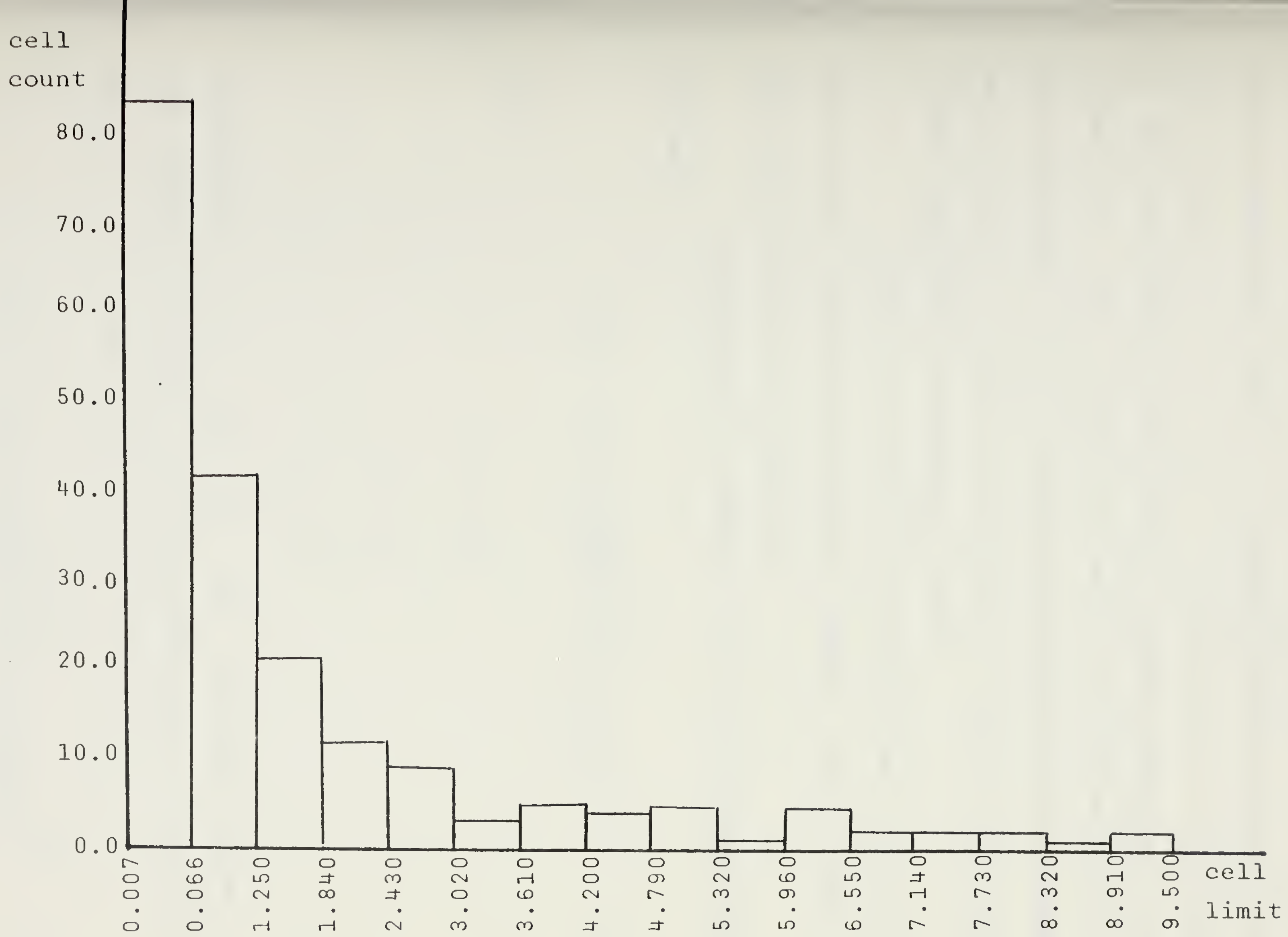

Fig. 8. Histogram for α

The goal is to choose values α, β and λ which minimize the expression (4.11). In the linear case, this can be done using elementary calculus by taking the partial derivatives with respect to α, β and λ, setting each of them equal to zero and solving the resulting linear equations (Normal equations). But the present problem is different because the expression (4.11) is highly nonlinear in the parameters, and indefinite in terms of convexity. This is why a nonlinear programming technique is used.

There are a few general approaches to the solution of the nonlinear estimation problem. One of them is the direct optimization approach.

Specifically, equation (4.9) is considered as follows:

$$z(p) = \frac{\rho_1 \rho_2 \varepsilon^2(p)}{\{1 + \rho_2 \varepsilon(p)\}\{1 + \rho_3 \varepsilon(p)\}} \tag{4.12}$$

where

$$\rho_1 = \frac{1}{\lambda}$$

$$\rho_2 = \frac{\alpha}{\lambda}$$

$$\rho_3 = \frac{\beta}{\lambda}$$

In equation (4.12), $z(p)$ and $\varepsilon(p)$ are known values, ρ_i are unknown variables. If it is rewritten as the sum of squared errors the objective function will be:

$$\sum_{i=1}^{N} \left\{ z(P_i) - \frac{\rho_1 \rho_2 \varepsilon^2(P_i)}{(1 + \rho_2 \varepsilon(P_i))(1 + \rho_3 \varepsilon(P_i))} \right\}^2 \tag{4.13}$$

Now nonlinear optimization problems can be stated such that

$$\text{Min} \sum_{i=1}^{N} \left\{ z(P_i) - \frac{\rho_1 \rho_2 \varepsilon^2(P_i)}{(1 + \rho_2 \varepsilon(P_i))(1 + \rho_3 \varepsilon(P_i))} \right\}^2$$

Subject to $\rho_1 \geq 0, \quad \rho_2 \geq 0, \quad \rho_3 \geq 0$

In order to perform the optimization, i.e. to solve this nonlinear optimization problem, the GRG package (Generalized Reduced Gradient) [6] was used; it is very convenient for this case. Ten runs were made with five different initial points, both with analytically computed derivatives and with numerically computed derivatives. The first four initial points were selected arbitrarily. The last one was picked such that:

$$\begin{aligned} \rho_1 &= \frac{1}{\hat{\lambda}} \\ \rho_2 &= \frac{z_{(1)}}{\hat{\lambda}} \\ \rho_3 &= \frac{z_{(50)}}{\hat{\lambda}} \end{aligned} \tag{4.14}$$

where $z_{(1)}$ is the smallest number in the sample, $z_{(50)}$ is the largest number in the sample, and $\hat{\lambda}$ is the natural logarithm of $p = 0.5$ divided by the median of the sample. Initial points are shown in Table II.

The optimization results of α, β, and ρ associated with ρ_1, ρ_2 and ρ_3 are tabulated in Table III and Table IV, respectively.

As it is seen from Tables III and IV, the program was terminated at different optimal points. Because the objective function is highly nonlinear and apparently indefinite (neither convex nor concave), it may sometimes have stopped at a local minimum rather than at a global optimum point. But if the results are compared to the 3-percentile results, the nonlinear least square estimates show much greater accuracy and seem more consistent.

TABLE II

INITIAL GUESS POINTS

# of Points	ρ_1	ρ_2	ρ_3
I	50.0	2.0	30.0
II	10.0	10.0	0.5
III	80.0	80.0	5.0
IV	1.0	1.0	1.0
V	According to the eq. (4.14)		

TABLE III

ESTIMATED VALUES OF ρ_1, ρ_2 AND ρ_3 BY USING NONLINEAR LEAST SQUARES

Real values $\rho_1 = 10.0$, $\rho_2 = 10.0$, $\rho_3 = 0.5$

# of Runs	Initial Point	ρ_1		ρ_2		ρ_3		Objective value	
		A	B	A	B	A	B	A	B
1	I	8.3141	8.3036	2,856.33	5,624.10	0.3333	0.3324	5.1063	5.1019
	II	8.3339	8.3086	1,080.65	8,044.10	0.3349	0.3327	5.1231	5.0998
	III	8.5549	8.3083	76.13	12,520.10	0.3507	0.3328	5.5045	5.0985
	IV	8.6341	8.3056	56.79	7,267.10	0.3564	0.3326	5.6517	5.1003
	V	8.3102	8.3254	36.76	7,956.10	0.3329	0.3344	5.1046	5.1002
2	I	10.1553	11.5808	76.51	15.03	0.4862	0.5915	3.7120	3.1776
	II	11.5494	11.5826	15.08	15.01	0.5886	0.5916	3.1784	3.1777
	III	10.2293	11.5819	62.99	15.02	0.4917	0.5915	3.6424	3.1776
	IV	12.5168	11.5808	10.15	14.96	0.6601	0.5919	3.2724	3.1777
	V	19.9871	11.5819	3.08	15.02	1.1656	0.5915	4.9374	3.1777

A. Computed without derivative.

B. Computed by using derivative.

TABLE III Cont.

# of Runs	Initial Point	ρ_1		ρ_2		ρ_3		Objective value	
		A	B	A	B	A	B	A	B
3	I	7.1926	6.9721	58.43	19,082.9	0.2468	0.2312	6.1643	5.3474
	II	6.9904	6.9714	968.39	10,597.6	0.2324	0.2315	5.3936	5.3496
	III	6.9434	6.9624	1,417.46	10,204.5	0.2283	0.2304	5.3860	5.3497
	IV	6.9703	6.9783	2,892.20	8,218.5	0.2309	0.2320	5.3610	5.3511
	V	22.7841	6.9718	1.27	14,336.6	1.1736	0.2314	19.5225	5.3483
4	I	11.3889	11.8592	1.35	1.27	0.4487	0.4709	7.4944	7.4888
	II	11.2345	11.4445	1.38	1.33	0.4414	0.4500	7.4977	7.4943
	III	11.2194	11.6732	1.38	1.29	0.4410	0.4614	7.4983	7.4912
	IV	11.4872	12.0535	1.33	1.24	0.4530	0.4802	7.4929	7.4877
	V	11.1144	11.6223	1.39	1.30	0.4340	0.4585	7.4998	7.4914
5	I	19.6997	19.6656	97.40	10.01	1.0449	1.5250	4.9284	3.8218
	II	14.3686	19.6123	12.18	10.07	1.4201	1.5210	3.8412	3.8218
	III	14.4505	19.6690	56.09	10.00	1.0825	1.5260	4.6573	3.8218
	IV	18.4853	19.6662	15.27	10.01	1.3263	1.5260	3.9070	3.8218
	V	14.7721	19.6902	9.93	9.97	1.5288	1.5280	3.8219	3.8218

TABLE III Cont.

# of Runs	Initial Point	ρ_1		ρ_2		ρ_3		Objective value	
		A	B	A	B	A	B	A	B
6	I	7.2217	10.0581	65.51	5.74	0.3573	0.5945	11.2795	9.7982
	II	9.8057	10.0363	6.15	5.78	0.3737	0.5927	9.8020	9.7982
	III	7.3314	10.0133	44.76	5.80	0.3662	0.5906	11.0600	9.7984
	IV	9.2639	10.0114	7.50	5.83	0.5293	0.5909	9.8415	9.7984
	V	7.8569	10.0520	19.16	5.74	0.4129	0.5939	10.4384	9.7982
7	I	8.6354	13.5421	58.55	4.08	0.3502	0.6801	8.4847	5.1221
	II	13.2375	13.5501	4.29	4.08	0.6573	0.6806	5.1251	5.1221
	III	13.2892	13.4448	4.24	4.15	0.6632	0.6742	5.1248	5.1225
	IV	13.1950	13.5141	4.33	4.10	0.6597	0.6783	5.1256	5.1222
	V	8.4124	13.5485	242.61	4.08	0.3344	0.6805	9.3879	5.1221
8	I	13.2670	13.3624	97.19	77.28	0.8558	0.8441	9.7124	9.7028
	II	13.5139	13.5059	62.85	64.32	0.8568	0.8567	0.6995	9.6994
	III	13.5926	13.5916	57.36	57.38	0.8638	0.8637	0.7014	9.7015
	IV	13.5569	13.4913	59.12	65.18	0.8597	0.8551	9.7012	9.6994
	V	58.6665	991.7600	1.94	0.85	4.2471	62.9137	12.8887	9.6995

TABLE III Cont.

# of Runs	Initial Point	ρ_1		ρ_2		ρ_3		Objective value	
		A	B	A	B	A	B	A	B
9	I	7.9547	7.9824	93.62	74.25	0.2188	0.2204	15.5568	15.5599
	II	7.9826	7.9771	81.27	74.82	0.2203	0.2197	15.5571	15.56
	III	8.0587	8.0218	56.27	68.97	0.2248	0.2238	15.5742	15.5646
	IV	7.9708	7.9685	87.76	82.04	0.2202	0.2197	15.5568	15.5575
	V	9.8247	7.9647	7.41	82.89	0.3251	0.2193	17.8068	15.5576
10	I	7.2072	6.9506	41.00	175.48	0.2354	0.2178	3.6436	3.5146
	II	6.9352	6.9389	200.00	205.08	0.2167	0.2172	3.5142	3.5114
	III	7.1424	6.9443	44.93	192.69	9.2331	0.2175	3.6199	3.5142
	IV	6.9583	6.9391	148.70	184.53	0.2183	0.2168	3.5160	3.5145
	V	7.0631	7.0049	70.92	102.40	0.2256	0.2216	3.5485	3.5249

TABLE IV

Actual values $\lambda = 0.1$, $\alpha = 1.0$, $\beta = 0.05$

# of Runs	Initial Point	λ		α		β	
		A	B	A	B	A	B
1	I	0.1202	0.1203	343.512	676.800	0.0400	0.0400
	II	0.1199	0.1203	129.591	968.150	0.0401	0.0401
	III	0.1168	0.1203	8.898	1506.926	0.0409	0.0400
	IV	0.1158	0.1204	6.577	874.957	0.0412	0.0385
	V	0.1203	0.1201	4.182	955.629	0.0400	0.0401
2	I	0.0984	0.0863	7.534	1.297	0.0478	0.0510
	II	0.0865	0.0863	1.306	1.296	0.0509	0.0510
	III	0.0977	0.0863	6.158	1.297	0.0480	0.0510
	IV	0.0798	0.0863	0.810	1.292	0.0527	0.0511
	V	0.0500	0.0863	0.154	1.296	0.0583	0.0511
3	I	0.1390	0.1434	8.123	392.550	0.0343	0.0331
	II	0.1430	0.1434	138.531	1520.067	0.0332	0.0332
	III	0.1440	0.1436	204.078	1465.586	0.0328	0.0330
	IV	0.1434	0.1433	414.903	1177.650	0.0331	0.0332
	V	0.0438	0.1434	0.055	2056.280	0.0515	0.0331
4	I	0.0878	0.0843	0.118	0.107	0.0394	0.0397
	II	0.0890	0.0873	0.123	0.116	0.0392	0.0393
	III	0.0891	0.0856	0.123	0.111	0.0393	0.0395
	IV	0.0870	0.0829	0.115	0.103	0.0394	0.0398
	V	0.0899	0.0860	0.125	0.112	0.0390	0.0394
5	I	0.0507	0.0508	4.944	0.509	0.0530	0.0775
	II	0.0695	0.0509	0.847	0.513	0.0988	0.0775
	III	0.0692	0.0508	3.882	0.508	0.0749	0.0775
	IV	0.0540	0.0508	0.826	0.509	0.0717	0.9775
	V	0.0676	0.0507	0.672	0.506	0.1034	0.0776

TABLE IV Cont.

# of Runs	Initial Point	λ		α		β	
		A	B	A	B	A	B
6	I	0.1384	0.0994	9.072	0.570	0.0494	0.0591
	II	0.1019	0.0996	0.628	0.576	0.0585	0.0590
	III	0.1363	0.0998	6.105	0.579	0.0499	0.0589
	IV	0.1079	0.0998	0.810	0.582	0.0571	0.0590
	V	0.1272	0.0994	2.439	0.571	0.0525	0.0590
7	I	0.1158	0.0738	6.780	0.301	0.0405	0.0502
	II	0.0755	0.0738	0.324	0.301	0.0496	0.0502
	III	0.0752	0.0743	0.319	0.309	0.0499	0.0501
	IV	0.0757	0.0739	0.328	0.308	0.0499	0.0501
	V	0.1188	0.0738	28.839	0.301	0.0397	0.0502
8	I	0.0753	0.0748	7.325	5.783	0.0645	0.0631
	II	0.0739	0.0740	4.647	4.763	0.0634	0.0634
	III	0.0735	0.0735	4.219	4.221	0.0635	0.0635
	IV	0.0737	0.0741	4.361	4.831	0.0634	0.0633
	V	0.0170	0.0010	0.033	0.0008	0.0724	0.0634
9	I	0.1257	0.1252	11.646	9.302	0.0275	0.0276
	II	0.1252	0.1253	10.181	9.374	0.0275	0.0275
	III	0.1240	0.1246	6.983	8.598	0.0278	0.0278
	IV	0.1254	0.1254	11.010	10.296	0.0276	0.0275
	V	0.1017	0.1255	0.754	10.407	0.0330	0.0275
10	I	0.1387	0.1438	5.688	25.170	0.0326	0.0313
	II	0.1441	0.1428	28.830	29.290	0.0312	0.0310
	III	0.1400	0.1440	6.290	27.747	0.0326	0.0313
	IV	0.1437	0.1441	21.370	26.590	0.0313	0.0312
	V	0.1415	0.1427	10.030	14.610	0.0310	0.0316

C. TEST PROCEDURE FOR PARAMETERS

In order to determine whether the model reasonably fits the data a simple test procedure is applied. It is to exhibit the simple comparison of the actual points $z(P)$ and the estimated values $\hat{z}(P)$, using the estimated parameters.

Basically $\hat{z}(P)$ can be obtained as follows:

$$\hat{x}(p) = -\frac{1}{\hat{\lambda}} \ell n(1-p) \tag{4.15}$$

Then if it is substituted in expression (2.37), $\hat{z}(P)$ will be

$$\hat{z}(P) = \frac{\hat{\alpha}\hat{x}^2(P)}{(1+\hat{\alpha}\hat{x}(P))(1+\hat{\beta}\hat{x}(P))} \tag{4.16}$$

In equation (4.16), all variables are known so estimated quantiles can be easily obtained.

In Fig. 9 actual values and estimated values are plotted. Also Fig. 10 displays residuals, which are the difference between actual and estimated values.

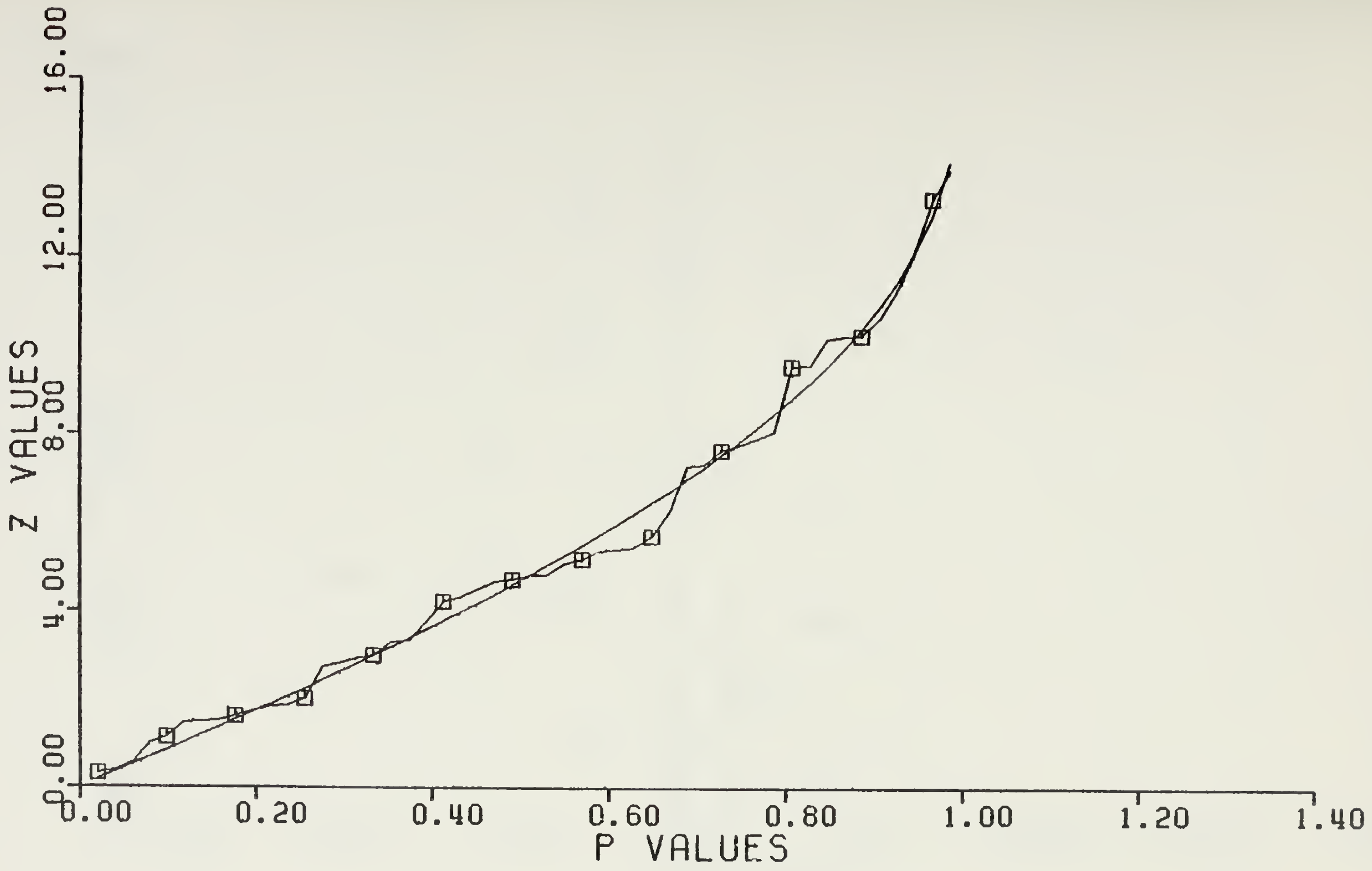

Fig. 9. Comparison of Actual Values and Estimated Values

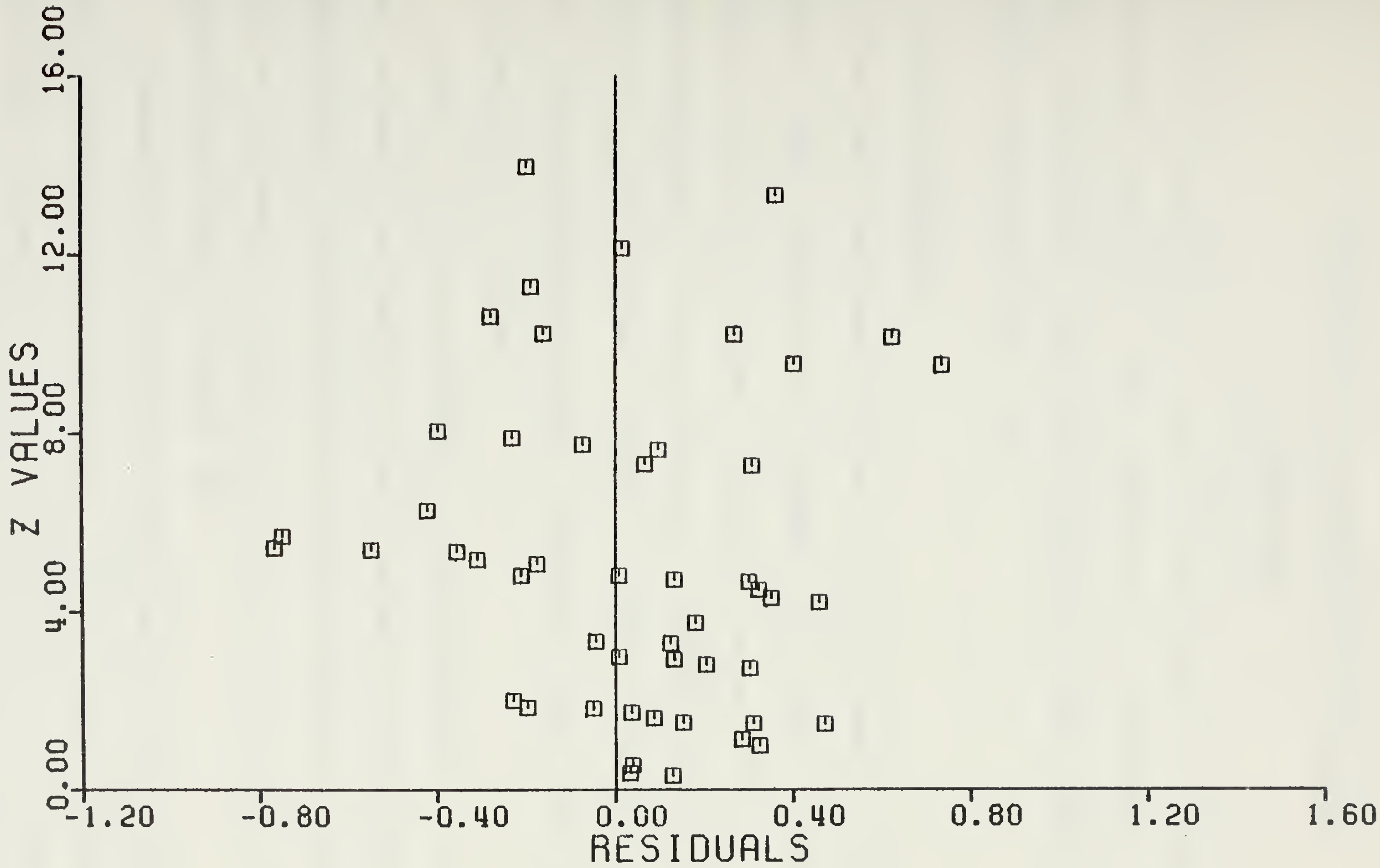

Fig. 10. Residuals of Fitting

V. NUMERICAL APPLICATIONS TO TWO SETS OF REAL DATA

In this section the two failure data sets studied come from these sources: Oral irrigator [7] and human life [8]. They are used for numerical examples in that these data are fitted using the model (2.6).

A. ORAL IRRIGATOR

The data used was obtained from the Commun. Statist.-Theor. meth., Colvert and Bordman [7]. The data was collected such that 100 oral irrigators were placed on the test was terminated at 700 time units. During the life test, 98 oral irrigators failed; another 2 oral irrigators survived. The ordered observed times to failure of the oral irrigators is tabulated in Table V.

Using the data of Table V, the parameters α, β, and λ are estimated by means of the 3-percentile approach. To demonstrate the differences between estimated values, various different combinations of p values were used. The results are shown in Table VI.

Examination of Table VI indicates that the 3-percentile approach may produce estimates having great differences for different percentile values, except here in the case of the first two combinations. Also the first two estimations seem to have accepatable limiting age. Moreoever, the last two combinations do not fit well, as judged from the residuals.

TABLE V

TIME TO FAILURE OF ORAL IRRIGATORS

1.75	7.02	7.58	9.76	15.02	15.57
17.39	19.55	22.47	23.24	23.96	25.05
32.44	36.87	42.76	43.14	46.95	56.33
58.99	59.08	60.37	61.01	77.86	86.45
88.50	103.06	104.34	105.85	117.46	120.11
122.28	122.61	129.31	130.42	137.57	142.27
142.98	148.29	150.79	151.21	155.62	157.93
160.72	169.79	186.26	197.60	224.83	233.64
242.07	256.86	260.77	261.68	277.99	283.95
288.94	295.48	314.76	316.06	332.07	339.46
362.61	369.47	370.74	491.06	403.39	414.78
426.71	459.62	455.84	457.94	466.61	468.64
469.09	476.42	481.41	481.82	488.15	490.06
493.67	494.38	503.72	508.93	509.01	418.32
532.29	534.62	545.23	547.41	558.41	571.10
585.52	589.11	592.93	607.15	623.15	647.91

TABLE VI

ESTIMATED VALUES OF α, β, λ BY USING 3-PERCENTILE

p_1	p_2	p_3	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$1/\beta$
.1	.5	.9	0.00153	0.00634	0.001028	972.76
.25	.5	.9	0.00158	0.00710	0.001018	982.31
.1	.5	.75	0.00239	0.01835	0.000228	4385.96
.25	.5	.75	0.00273	0.04514	0.000077	12987.01

Next the nonlinear optimization approach was applied to estimate parameters α, β and λ associated with ρ_1, ρ_2 and ρ_3, both using the derivative and without derivative by taking into account 98 ordered values. The results are tabulated in Tables VII and VIII for ρ_1, ρ_2, ρ_3 and λ α, β respectively. Some initial points in Table II are used. Table VII indicates that the estimates obtained from the nonlinear estimation method are more consistent than those from the 3-percentile approach. Also, the boundary points, $1/\beta$, seem reasonable. Finally, comparison of the actual values of $z(p)$ and estimated values of $\hat{z}(p)$ indicate that the fitting is reasonable. The residuals of fitted model and the comparison of the actual and estimated values are plotted in Fig. 11 and Fig. 12.

B. HUMAN LIFE (MORTALITY) DATA

The data used in this example was obtained from the 1969-71 life table [8] for white females in the United States. The table has been prepared from a history of 100,000 persons: the number of surviving, and the number dying has been given for each age interval. Look at Table IX.

1. Estimation by Using 3-Percentile Approach

The life data is first fitted to the model (2.37) by using the 3-percentile approach. Certain pth percentile values are selected and used in the estimation process. These results are tabulated in Table X. Investigation of the last eight combinations of percentiles in these tables

TABLE VII

ESTIMATED PARAMETERS FOR ORAL IRRIGATORS

Initial Points	ρ_1		ρ_2		ρ_3		Objective Value	
	A	B	A	B	A	B	A	B
I	1374.10	1417.50	1.629	1.578	1.530	1.577	54040.	59016.
II	460.27	1415.75	962.574	1.580	0.411	1.575	139009.	59016.
III	1359.83	1418.23	1.642	1.577	1.511	1.577	59052.	59016.
IV	1384.93	459.85	1.610	3354.840	1.538	0.411	59030.	139571.
V	713.32	1419.54	357.163	1.576	0.035	1.579	337274.	59016.

TABLE VIII

Initial Points	$\hat{\lambda}$		$\hat{\alpha}$		$\hat{\beta}$		$1/\hat{\beta}$	
I	0.000722	0.000705	0.001176	0.001113	0.001104	0.001112	905.0	898.7
II	0.002172	0.000706	2.089900	0.001116	0.000894	0.001112	1118.0	898.8
III	0.000736	0.000705	0.001200	0.001112	0.001112	0.001112	899.0	898.8
IV	0.000722	0.002172	0.001162	7.293600	0.001110	0.000893	901.0	1118.8
V	0.001401	0.000704	0.500700	0.001110	0.001310	0.001112	724.0	899.1

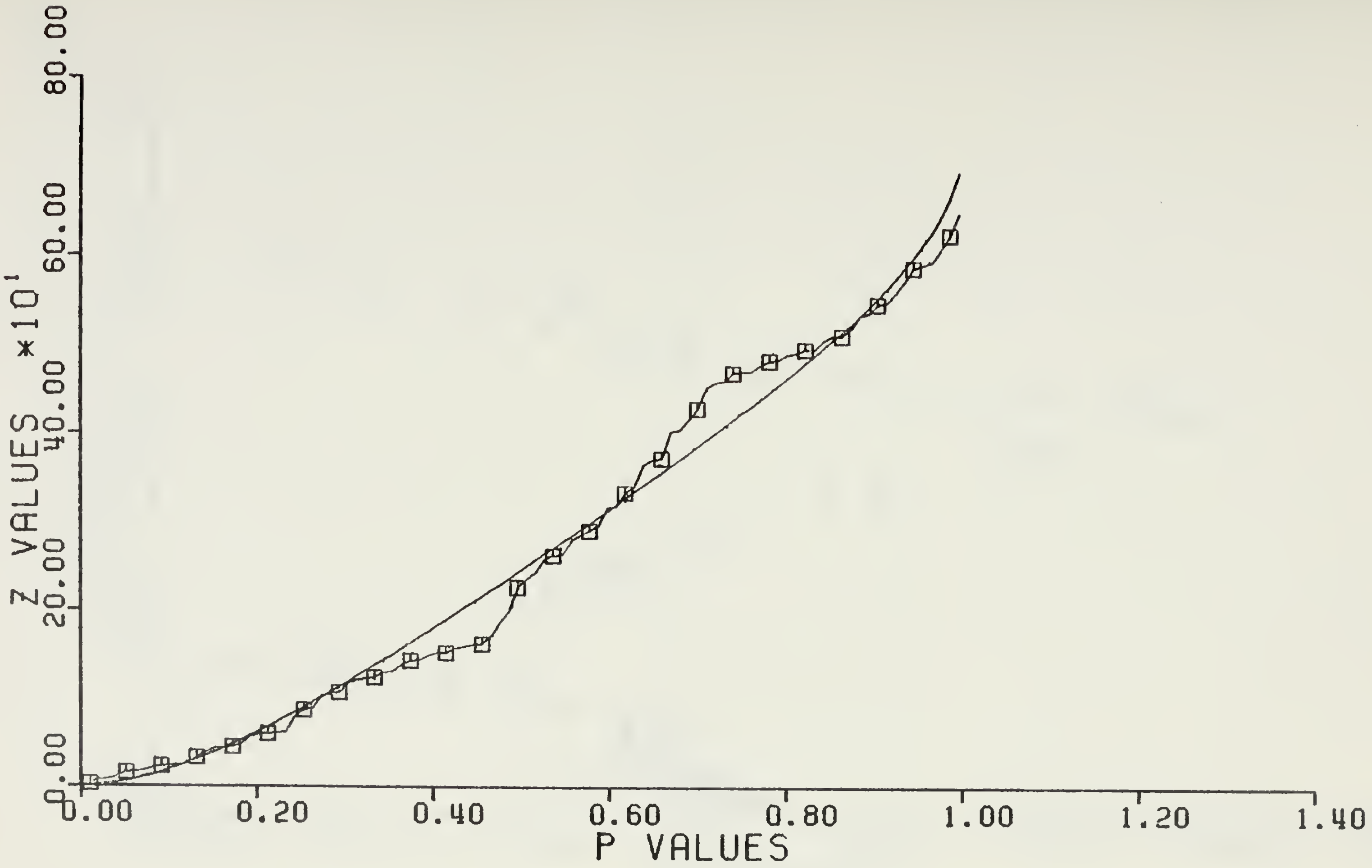

Fig. 11. Comparison of Actual Values and Estimated Values

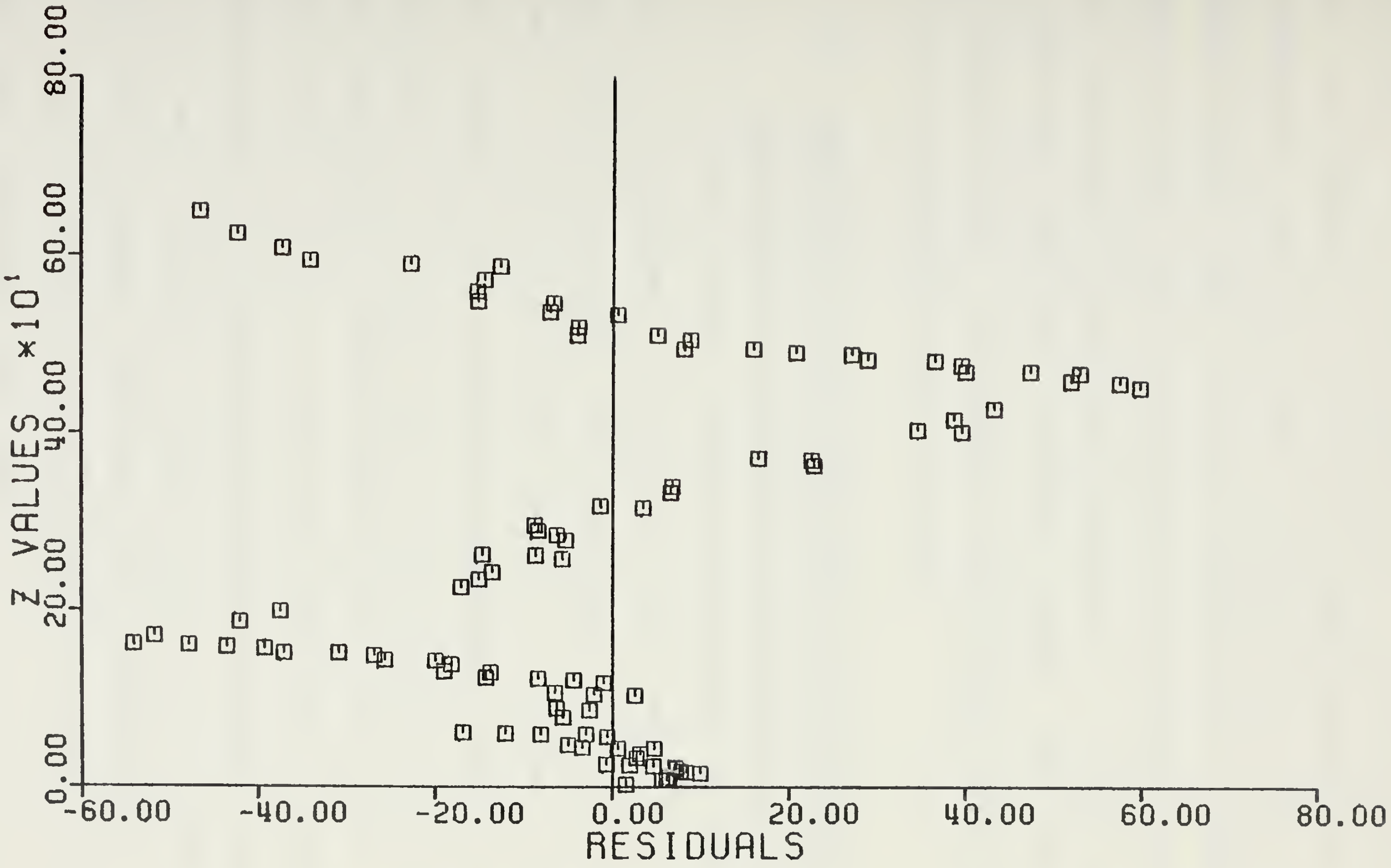

Fig. 12. Residuals of Fitting

indicates that the limiting age, $1/\beta$, is approximately 110, which is reasonable. However, the first combination of percentiles gives a good fit in terms of the sum of the squared errors. But for these limiting age is reduced to about 95 years. This apparently means that the fit of the model does not represent the age above 95. Thus it can be said that there is a trade-off between $1/\beta$ and the sum of squared errors. The present model simply does not seem to fit the mortality data very well.

2. Estimation By Using Nonlinear Least Squares, and Application of a Nonlinear Programming Algorithm

In the previous section we discussed the fact that one of the problems encountered in the 3-percentile estimation procedure was the high uncertainty of fitting, as measure by the sum of squared errors. In order to reduce this variability, a nonlinear estimation process is applied. Previously it was noted that the GRG package can be used with analytically computed derivatives and without derivatives. If it is used with analytically computed derivatives, it is necessary to provide another subprogram by the user. Otherwise the GRG package will provide the derivative, computed numerically and automatically.

The derivative of the objective function (4.13) is the normal equations such that

TABLE IX

MORTALITY DATA FOR WHITE FEMALES, 1969-71

Time Interval	# of Survivors	# of Dying	Time Interval	# of Survivors	# of Dying
0- 1	100,000	1532	27-28	97,165	70
1- 2	99,468	100	28-29	97,095	73
2- 3	98,368	65	29-30	97,022	77
3- 4	98,303	54	30-31	96,945	81
4- 5	98,249	46	31-32	96,864	87
5- 6	98,203	39	32-33	96,777	93
6- 7	98,164	35	33-34	96,684	101
7- 8	98,129	32	34-35	96,583	109
8- 9	98,097	29	35-36	96,474	118
9-10	98,068	26	36-37	96,356	128
10-11	98,042	24	37-38	96,228	141
11-12	98,018	24	38-39	96,087	153
12-13	97,994	25	39-40	95,932	170
13-14	97,969	30	40-41	95,762	185
14-15	97,939	37	41-42	95,577	201
15-16	97,902	45	42-43	95,376	220
16-17	97,857	54	43-44	95,156	242
17-18	97,803	60	44-45	94,914	263
18-19	97,743	62	45-46	94,649	291
19-20	97,681	63	46-47	94,358	317
20-21	97,618	63	47-48	94,041	344
21-22	97,555	63	48-49	93,697	372
22-23	97,492	63	49-50	93,325	401
23-24	97,429	64	50-51	92,924	433
24-25	97,365	66	51-52	92,491	469
25-26	97,299	66	52-53	92,022	506
26-27	97,233	68	53-54	91,516	546

TABLE IX Cont.

Time Interval	# of Survivors	# of Dying	Time Interval	# of Survivors	# of Dying
54-55	90,970	587	82- 83	41,215	3,586
55-56	90,383	632	83- 84	32,629	3,589
56-57	89,751	680	84- 85	34,040	3,550
57-58	89,071	730	85- 86	30,490	3,495
58-59	88,341	781	86- 87	26,995	3,425
59-60	87,560	834	87- 88	23,570	3,286
60-61	86,726	911	88- 89	20,284	3,072
61-62	85,835	953	89- 90	17,212	2,806
62-63	84,882	1,022	90- 91	14,406	2,531
63-64	83,860	1,098	91- 92	11,875	2,262
64-65	82,762	1,183	92- 93	9,613	1,982
65-66	81,579	1,275	93- 94	7,631	1,694
66-67	80,304	1,375	94- 95	5,937	1,411
67-68	78,929	1,486	95- 96	4,526	1,145
68-69	77,443	1,607	96- 97	3,381	905
69-70	75,836	1,735	97- 98	2,476	696
70-71	74,101	1,862	98- 99	1,780	524
71-72	72,239	1,993	99-100	1,256	384
72-73	70,246	2,141	100-101	872	277
73-74	68,105	2,312	101-102	595	195
74-75	65,793	2,503	102-103	400	135
75-76	63,290	2,693	103-104	265	92
76-77	60,597	2,872	104-105	173	61
77-78	57,725	3,039	105-106	112	41
78-79	54,686	3,187	106-107	71	26
79-80	51,499	3,317	107-108	45	17
80-81	48,182	3,434	108-109	28	11
81-82	44,748	3,533	109-110	17	6

TABLE X

ESTIMATED VALUES OF α, β, λ FOR MORTALITY DATA USING 3-PERCENTILE APPROACH

Percentile Values	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$1/\hat{\beta}$
22 56 92	0.000674	0.1190	0.010550	94.7
20 56 92	0.000550	0.0404	0.010520	95.0
17 56 92	0.000508	0.0302	0.010550	94.7
16 56 92	0.000440	0.0199	0.010550	94.7
17 56 109	0.000862	0.1676	0.009069	110.2
18 56 109	0.000886	0.2389	0.009068	110.2
19 56 109	0.000908	0.3708	0.009067	110.2
17 56 110	0.000876	0.1805	0.008991	111.2
18 56 110	0.000899	9.2613	0.008989	111.2
19 56 110	0.000920	0.4188	0.008988	111.2
20 56 110	0.000941	0.9070	0.008988	111.2

$$\frac{\partial S}{\partial \rho_1} = -2 \sum_{i=1}^{N} \left\{ \frac{\rho_2 \varepsilon^2(P_i)}{(1+\rho_2\varepsilon(P_i))(1+\rho_3\varepsilon(P_i))} \right\}$$

$$\times \left\{ z(P_i) - \frac{\rho_1\rho_2\varepsilon^2(P_i)}{(1+\rho_2\varepsilon(P_i))(1+\rho_3\varepsilon(P_i))} \right\} \qquad (4.17)$$

$$\frac{\partial S}{\partial \rho_2} = -2 \sum_{i=1}^{N} \left\{ \frac{\rho_1\varepsilon^2(P_i)(1+\rho_2\varepsilon(P_i)) - \varepsilon(P_i)\,\rho_1\rho_2\varepsilon^2(P_i)}{(1+\rho_2\varepsilon(P_i))^2\,(1+\rho_3\varepsilon(P_i))} \right\}$$

$$\times \left\{ z(P_i) - \frac{\rho_1\rho_2\varepsilon^2(P_i)}{(1+\rho_2\varepsilon(P_i))(1+\rho_3\varepsilon(P_i))} \right\} \qquad (4.18)$$

$$\frac{\partial S}{\partial \rho_3} = +2 \sum_{i=1}^{N} \left\{ \frac{\rho_1\rho_2\varepsilon^2(P_i)}{(1+\rho_2\varepsilon(P_i))^2\,(1+\rho_3\varepsilon(P_i))} \right\}$$

$$\times \left\{ z(P_i) - \frac{\rho_1\rho_2\varepsilon^2(P_i)}{(1+\rho_2\varepsilon(P_i))(1+\rho_3\varepsilon(P_i))} \right\} \qquad (4.19)$$

N is 110 in the case of human mortality data.

The results of estimation were shown in Tables XI and XII for ρ_i and α, β, λ respectively. Also Fig. 13 and Fig. 14 demonstrate the actual and estimated values and residuals of fitting.

Examination of Table XI and Table XII indicates that the objective function values (the sum of the squared errors) of the fitting are smaller than the objection function values of 3-percentile approach. Also the estimated

parameter values $\hat{\beta}$ and $\hat{\lambda}$ associated with ρ_1 and ρ_3 are very close to each other. But estimated values of $\hat{\alpha}$ show considerable difference for the various initial values. However, there are two important facts to notice. First, when α increases significantly, the objective function values remain almost constant. That is, it is not effected significantly on the objective function values in this case. Second, this analytical model (2.6) does not represent deaths beyond the age 95. However, modifications can be made in the model (2.6) for this kind of difficulty. Some ideas will be discussed in a later section.

3. Model Modifications

In the previous section it was noted that the model (2.6) has relatively great errors and does not accurately describe probability of death at age greater than 95 in the human life example. This means that the hazard function $h_z(z)$ increases too rapidly in the wearout period. If it is possible to slow the rate of increase of hazard perhaps a better result can be obtained.

In Section II, it was stated that the function $R(x)$ describes the wearout period in bath tub type curves:

$$R(x) = \frac{1}{1 + \beta x} , \qquad \beta > 0$$

If a new parameter, γ, is defined which is between 0 and 1 and $R'(x)$ is now defined to be the γ^{th} power of $R(x)$:

$$R'(x) = \frac{1}{(1 + \beta x)^{\gamma}}, \qquad \beta > 0, \quad 0 < \gamma < 1 \qquad (4.20)$$

then $R'(x)$ will provide for a slowly increasing, rather than rapidly increasing, in wearout period. With this revision the model (2.6) now becomes

$$z = G(x) = xL(x) \cdot R'(x) \qquad (4.21)$$

or

$$z = \frac{\alpha x^2}{(1 + \alpha x)(1 + \beta x)^{\gamma}} \qquad (4.22)$$

where $\alpha > 0$, $\beta > 0$ and $0 < \gamma < 1$.

However, after we put another variable in the model, it will be too difficult to handle in the previous estimation technique for obtaining values of the parameters α, β, λ and γ because of nonlinearity and indefiniteness of the expression (4.22).

For this reason, γ will be assumed constant in the previous estimation procedure. Then it will be computationally convenient. Actually, the estimation procedure does not require any change. The nonlinear least square estimation approach will simply be used for different values of γ. Table XIII and Fig. 15 demonstrate the parameters estimated and the resulting objective function values. From Table XIII, it can be easily seen that the new result has a smaller sum of squared errors. The best value of $\hat{\gamma}$ seems to be near 0.95.

TABLE XI

ESTIMATED PARAMETERS FOR HUMAN LIFE BY USING SAME INITIAL POINTS

Initial Points	ρ_1		ρ_2		ρ_3		Objective value	
	A	B	A	B	A	B	A	B
I	1198.37	1198.27	514.70	514.70	12.3319	12.3301	4335.89	4335.89
II	1120.16	1119.94	8578.22	8591.20	11.4797	11.4780	4426.37	4426.38
III	1600.10	1356.57	105.48	188.34	16.6541	14.0573	4340.41	4287.21
IV	1189.13	1352.87	570.01	190.40	12.2301	14.0129	4342.65	4287.21
V	1352.11	1353.58	188.53	188.42	14.0054	14.0209	4287.30	4287.28

A. Computed without derivative.

B. Computed by using derivative.

TABLE XII

ESTIMATED PARAMETERS α, β, λ FOR HUMAN LIFE BY USING SAME INITIAL VALUES

Initial Points	$\hat{\lambda}$		$\hat{\alpha}$		$\hat{\beta}$		$1/\hat{\beta}$	
	A	B	A	B	A	B	A	B
I	0.000834	0.000834	0.42950	0.42953	0.01029	0.01029	97.1	97.2
II	0.000892	0.000893	7.65803	7.67482	0.01024	0.01025	97.5	97.5
III	0.000625	0.000737	0.06592	0.13882	0.01040	0.01036	96.0	96.5
IV	0.000841	0.000739	0.47968	0.14073	0.01028	0.01033	97.2	96.8
V	0.000740	0.000738	0.13948	0.13928	0.01035	0.01035	96.5	96.5

A. Estimation with derivative.

B. Estimation by using derivative.

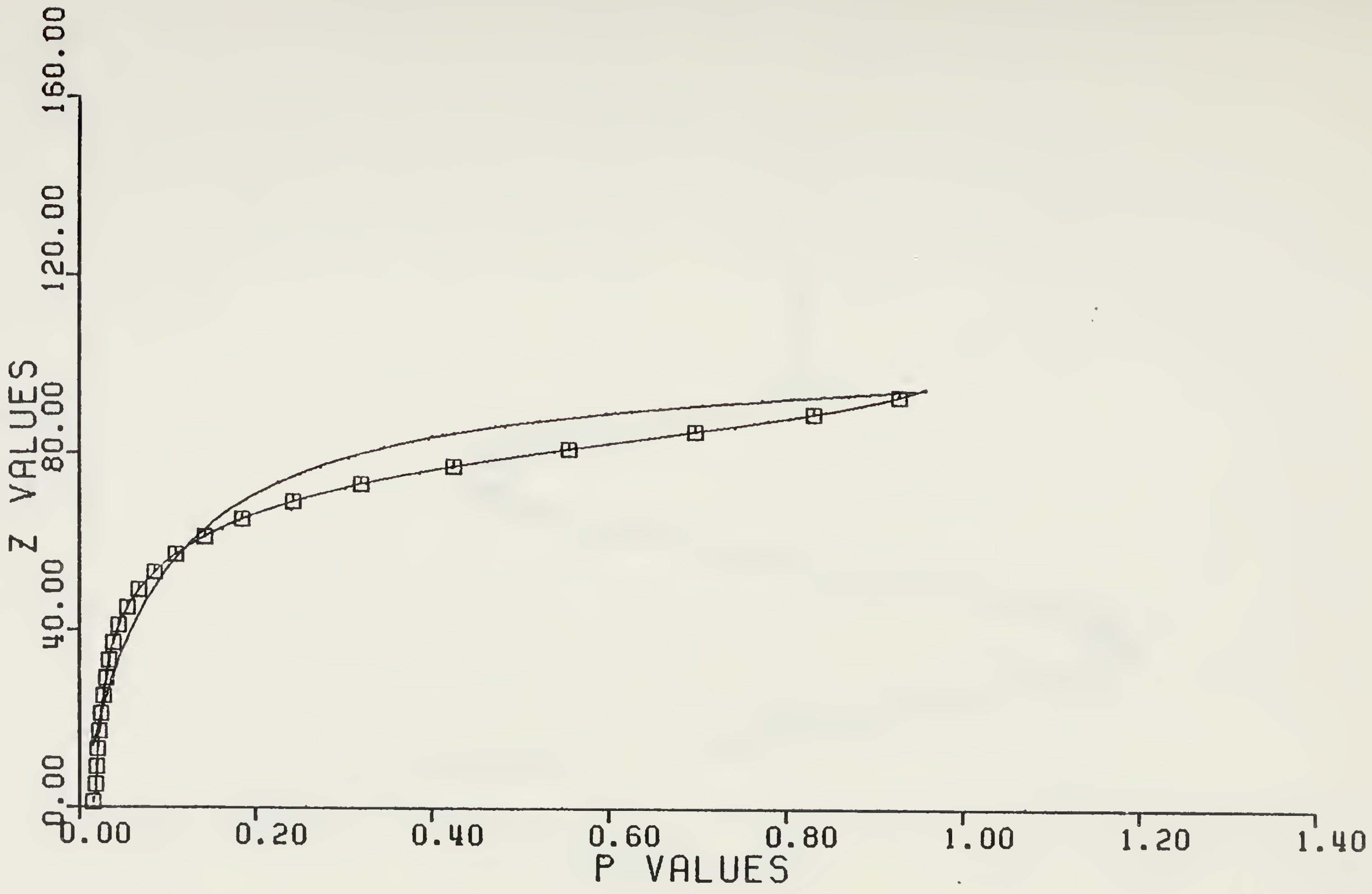

Fig. 13. Comparison of Actual Values and Estimated Values

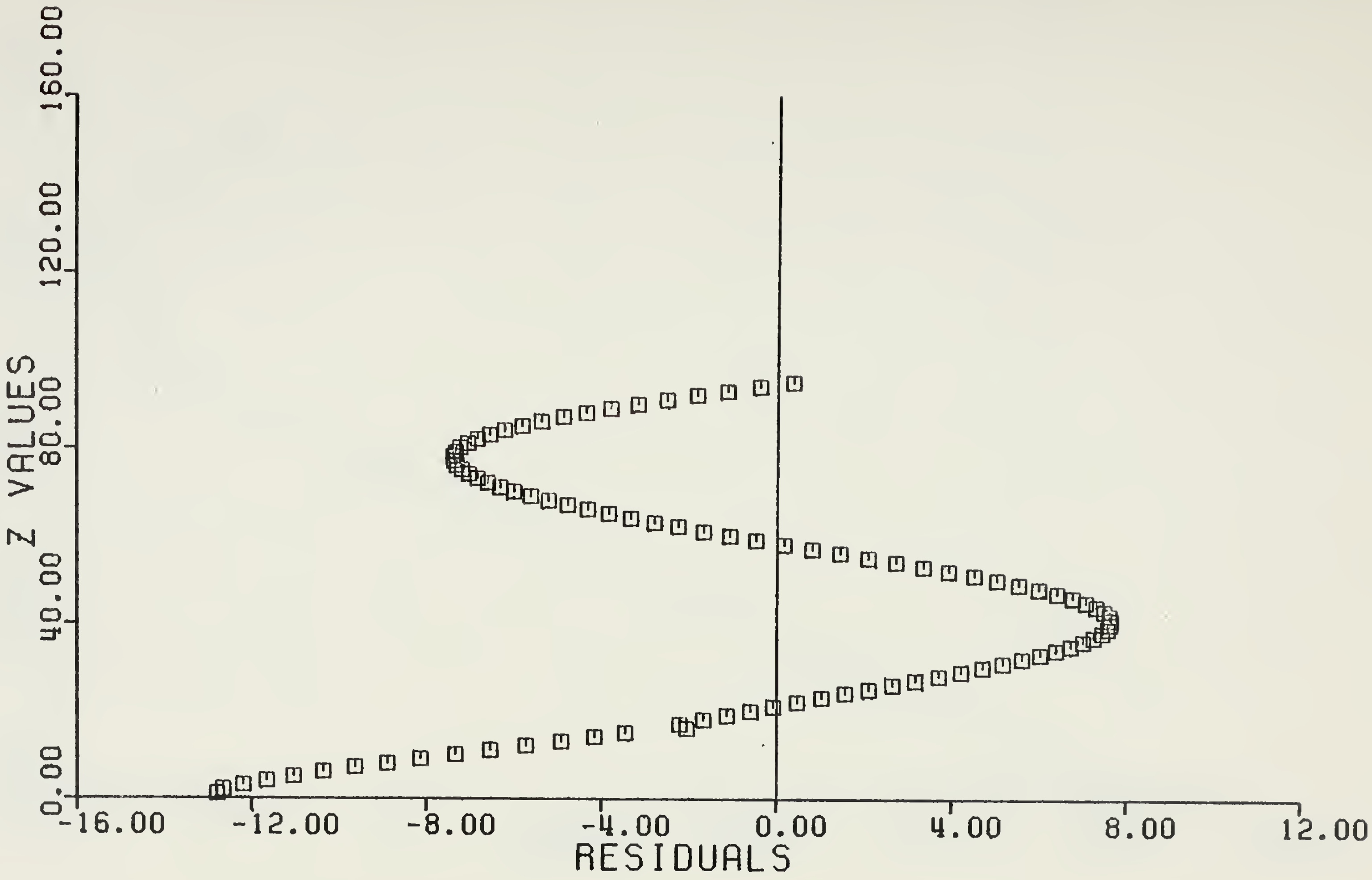

Fig. 14. Residuals of Fitting

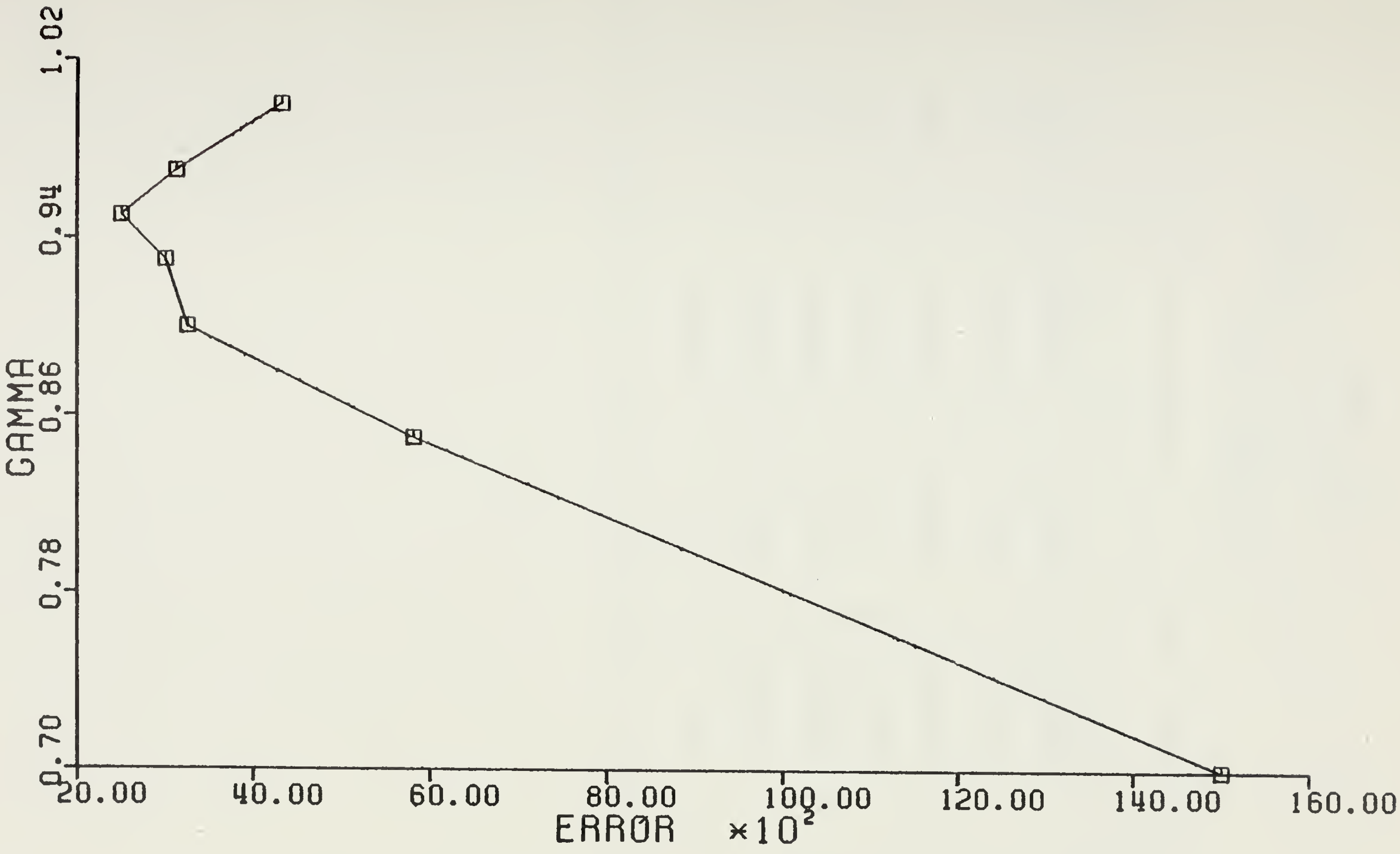

Fig. 15. Plot of Different γ Values vs Objective Values

TABLE XIII

ESTIMATION VALUES OF α, β, λ USING CONSTANT γ

# of Run	γ	λ	α	β	Objective Value
1	1.000	0.000834	0.42950	0.01029	4,335.89
2	0.97	0.000512	0.04051	0.01182	3,119.85
3	0.95	0.000363	0.01928	0.01314	2,494.10
4	0.93	0.000740	1.06371	0.01352	2,999.11
5	0.90	0.000685	1.81283	0.01554	3,267.47
6	0.85	0.000592	5.74169	0.02031	5,099.37
7	0.7	0.000319	3.17467	0.04580	15,310.06

C. CONCLUSIONS

Simple analytical hazard models have been developed and fitted to situations (data) that exhibit bath tub shaped hazard functions. That is, failure rates may be high at early ages ("infant mortality"), constant at intermediate ages, and high again for later ages ("wearout"). The procedure emphasizes representations of the inverse distribution function; simulation is thus facilitated.

The failure time distributions so derived should be useful in analyzing maintenance and replacement policies.

A least squares technique for fitting the hazard models to data are suggested and applied.

LIST OF REFERENCES

1. Tukey, J.W., Usable Resistant/Robust Techniques of Analysis (p. 11). Proc. of the First ERDA Statistical Symposium. Battle Pacific Northwest Laboratories, Richland, Washington, 1976.

2. Parzen, M., Nonparametric Statistical Data Modelling. Paper presented at National Meeting American Statistical Association, San Diego, August 1978.

3. Gaver, D.P., Lavenberg, S., Price, T., Exploratory analysis of access path length data for a data base management system. IBM J. of Research Development, Vol. 20 (pp. 449-464), 1976.

4. Gaver, D.P. and Chu, B., Estimating equipment and system availability by use of the Jackknife, ERPI Tech. Rept.; also Naval Postgraduate School Technical Report NPS55-77-4, 1977.

5. Hammersley, J. and Handscomb, D.C., Monte Carlo Methods, John Wiley, 1964.

6. Technical Memorandum CIS-75-02, GRG User's Guide, Lasdon, L.S., Waren, A.D., Ratner, M.W., and Jain, A., 1975.

7. Colvert, R.E., and Boardman, T.J., Estimation in the Piece-wise Constant Hazard Rate Model, Commun. Statist. -Theor. meth., A5(11), 1013-1029, 1976.

8. United States Life Table for White Females, 1969-71.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	2
2.	Library, Code 0142 Naval Postgraduate School Monterey, California 93940	2
3.	Department Chairman, Code 55Zo Department of Operations Research Naval Postgraduate School Monterey, California 93940	1
4.	Professor D. P. Gaver, Code 55Gv Department of Operations Research Naval Postgraduate School Monterey, California 93940	2
5.	Professor F. Russell Richards, Code 55Rh Department of Operations Research Naval Postgraduate School Monterey, California 93940	1
6.	Lt Mustafa Acar, Turkish Navy Kadimehmet mah. Akide Sok. No. 7/1, Kasimpasa Istanbul, TURKEY	1
7.	Deniz Kuvvetleri Komutanliḡi Bakanliklar, Ankara, Turkey	5
8.	Deniz Harb Okulu Komutanliḡi Heybeliada, Istanbul, TURKEY	1
9.	Orta-Dogu Teknik Universitesi Yon-Eylem Arastirmasi Bolumu Ankara, TURKEY	1
10.	CPT Michael P. Mueller Heerseflugabwehrscdiule Schleswiger Chaussee 239 Rendsburg West Germany	1

Thesis 177783
A1735 Acar
c.1
Analytical hazard representations for use in reliability, mortality and simulation studies.

9 AUG 79 25274
7 MAR 80 25286
29 DEC 80 26405
20 APR 84 29342

Thesis 177783
A1735 Acar
c.1
Analytical hazard representations for use in reliability, mortality and simulation studies.

thesA1735
Analytical hazard representations for us
3 2768 000 98921 4
DUDLEY KNOX LIBRARY